- Calculators are NOT allowed in the exam.

Question 1. *[10 pts] Let $f : [a, b] \to \mathbb{R}$ be a one-to-one and continuous function. If $f(a) < f(b)$, show that f is increasing.*

Question 2. *[20 pts] Let Ω be the domain in the xy-plane bounded by the x-axis and one arc of the cycloid*

$$(x, y) = (\theta - \sin\theta, 1 - \cos\theta), \theta \in [0, 2\pi].$$

Find the volume of the solid

$$\{(x, y, z) \in \mathbb{R}^3 : 0 < z < y^2, (x, y) \in \Omega\}.$$

Question 3. *[20 pts] Suppose that the cubic equation $a_0 + b_0 x + c_0 x^2 + d_0 x^3 = 0$ has three distinct roots $\alpha_0 < \beta_0 < \gamma_0$. Given $\epsilon > 0$, prove that there exists $\delta > 0$ such that for every $(a, b, c, d) \in \mathbb{R}^4$ with*

$$(a - a_0)^2 + (b - b_0)^2 + (c - c_0)^2 + (d - d_0)^2 < \delta^2,$$

the equation $a + bx + cx^2 + dx^3 = 0$ has three roots α, β, and γ satisfying

$$(\alpha - \alpha_0)^2 + (\beta - \beta_0)^2 + (\gamma - \gamma_0)^2 < \epsilon^2.$$

Question 4. *[20 pts] For each $\epsilon \in (0, 1)$, let*

$$S_\epsilon = \left\{(x, y, z) \in \mathbb{R}^3 : \epsilon < \sqrt{x^2 + y^2} < 1, -1 + \epsilon < z < 1\right\}.$$

Evaluate

$$\lim_{\epsilon \to 0+} \iiint_{S_\epsilon} \frac{z}{\sqrt{x^2 + y^2}\,\ln(1 + x^2 + y^2)}\,dx\,dy\,dz.$$

Question 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded and continuous function. For every $k \in \mathbb{N}$ and $x \in \mathbb{R}$, define*

$$f_k(x) = \min_{y \in \mathbb{R}} (f(y) + k|y - x|).$$

(a) [10 pts] Show that $f_k(x)$ is well-defined (i.e., the minimum does exist).

(b) [10 pts] Show that $f_k : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous.

(c) [10 pts] Prove that on every finite interval, f_k converges to f uniformly as $k \to \infty$.

試題隨卷繳回