※ 注意：請於試卷上「非選擇題作答區」標明大題及小題題號，並依序作答。

1. Let $\{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$. Suppose that a linear transformation $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is defined by $T(x, y, z) = (2x + y, 2y + z, 2z)$.
   (a) Write down the matrix of T relative to the standard basis. **(5 points.)**
   (b) Write down the matrix of T relative to the ordered basis $\{e_3, e_2, e_1\}$. **(5 points.)**
   (c) Find a matrix P such that
   $$P^{-1}\begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix} P = \begin{pmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 1 & a \end{pmatrix}$$
   for all real numbers a. **(5 points.)**
   (d) Prove that for any given $n \times n$ matrix A, there is a matrix Q such that
   $$Q^{-1}AQ = A^t.$$
   (5 points.)
   (e) Let
   $$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$
   Find a matrix Q such that $Q^{-1}AQ = A^t$. **(5 points.)**
2. Let A and B be two square matrices over $\mathbb{C}$. Prove that the set of eigenvalues of AB is the same as the set of eigenvalues of BA. **(15 points.)**
3. Let θ be a real number that is not an integer multiple of π (so that $\sin\theta \neq 0$). For a positive integer n, let $A_n = (a_{ij})$ be the $n \times n$ matrix defined by
   $$a_{ij} = \begin{cases} 0, & \text{if } |i-j| > 1, \\ 1, & \text{if } |i-j| = 1, \\ 2\cos\theta, & \text{if } i = j. \end{cases}$$
   Prove that $\det A_n = \dfrac{\sin(n+1)\theta}{\sin\theta}$. **(15 points.)**
4. Let
   $$A = \begin{pmatrix} 3 & -2 & -2 \\ -2 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}.$$
   (a) Find the maximum of X^tAX among all $X \in \mathbb{R}^3$ subject to $X^tX = 1$. Give an example of X that attains the maximum. **(8 points.)**
   (b) Find the minimum of $\operatorname{tr}(Y^tAY)$ among all 3×2 matrices Y over $\mathbb{R}$ subject to $Y^tY = I_2$. Give an example of Y that attains the minimum. **(7 points.)**
5. Let $V = \{a_0 + a_1x + a_2x^2 : a_j \in \mathbb{R}\}$ be the vector space of all polynomials of degree 2 or less over $\mathbb{R}$. Define three linear functionals on V by
   $$f_1(p) = \int_0^1 p(x)\,dx, \qquad f_2(p) = \int_0^2 p(x)\,dx, \qquad f_3(p) = \int_{-1}^0 p(x)\,dx.$$
   Find a basis $\mathcal{B}$ for V such that $\{f_1, f_2, f_3\}$ is its dual basis. **(15 points.)**
6. Find all possible Jordan forms for 7×7 real matrices having $x^2(x-1)^2$ as minimal polynomial. **(15 points.)**

試題隨卷繳回