※ 注意：請於試卷上「非選擇題作答區」標明題號並依序作答。

(1) (25 points) Let $A = \begin{pmatrix} -2 & -1 & 2 & -1 \\ 11 & 7 & -1 & 5 \\ -8 & -2 & 6 & -2 \\ 4 & 1 & -2 & 3 \end{pmatrix}$. Find the Jordan canonical form of $A$. Compute $\exp(tA)$ and derive the general solution to $x'(t) = A\,x(t)$, where $x(t)$ is a 4-dimensional column vector.

(2) (25 points) Let $A$ and $B$ be any $n \times n$ complex matrices. Show that $\exp(A)\exp(B) = \exp(A+B)$ if $A$ and $B$ commute.
Hint: You may consider the norm $\|C\| = \max\{\, |C_{ij}| \,\}$ for $C = (C_{ij}) \in M_n(\mathbb{C})$ and the remainder term

$$R_p = \sum_{i=0}^{2p} \frac{(A+B)^i}{i!} - \sum_{j=0}^{p} \frac{A^j}{j!} \sum_{k=0}^{p} \frac{B^k}{k!}.$$

(3) (25 points) Show that

$$\det \begin{pmatrix} X_0 & X_1 & X_2 & \dots & X_{n-1} \\ X_{n-1} & X_0 & X_1 & \dots & X_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_1 & X_2 & X_3 & \dots & X_0 \end{pmatrix} = \prod_{j=0}^{n-1} (\sum_{k=0}^{n-1} \zeta^{jk} X_k)$$

where $\zeta$ is a primitive $n$-th root of unity.

Hint: You may first compute, for example, $\begin{pmatrix} X_0 & X_1 & X_2 & X_3 \\ X_3 & X_0 & X_1 & X_2 \\ X_2 & X_3 & X_0 & X_1 \\ X_1 & X_2 & X_3 & X_0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 & \zeta^3 \\ 1 & \zeta^2 & \zeta^4 & \zeta^6 \\ 1 & \zeta^3 & \zeta^6 & \zeta^9 \end{pmatrix}$.

(4) (25 points) Let $V$ and $W$ be vector spaces over the same field $F$, and let $T : V \to W$ be a linear transformation. Let $\mathcal{B}(V)$ and $\mathcal{B}(W)$ denote the spaces of bilinear forms on $V$ and $W$ respectively. For any $H \in \mathcal{B}(W)$, define $\widehat{T}(H) : V \times V \to F$ by $\widehat{T}(H)(x, y) = H(T(x), T(y))$ for any $x, y \in V$.
(i) Show that $\widehat{T}(H)$ is a bilinear form on $V$.
(ii) Show that if $T$ is an isomorphism, then so is $\widehat{T} : \mathcal{B}(W) \to \mathcal{B}(V)$.

試題隨卷繳回