※ 注意：請於試卷上「非選擇題作答區」標明題號並依序作答。

1. [20%] Prove L'Hospital's rule in the following case: Suppose $f(x), g(x)$ are differentiable with continuous derivatives $f'(x), g'(x)$ on the interval $(-1, 1)$ such that $f(0) = g(0) = 0$ and $g'(0) \neq 0$. Then
$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}.$$

2. (a) [5%] Give an example of differentiable functions $f_n(x), n \geq 0$, on $(-1, 1)$ such that $\sum_{n=0}^{\infty} f_n(x)$ converges to a function on $(-1, 1)$ which is *not* differentiable. You need to justify your answer.

   (b) [15%] Suppose the power series $\sum_{n=0}^{\infty} a_n x^n$ converges to a function $f(x)$ on the interval $(-1, 1)$. Show that $f(x)$ is differentiable with derivative $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ on $(-1, 1)$.

3. [20%] Let $f(x, y)$ be a function on $\mathbb{R}^2$ whose partial derivatives of any order are continuous. Suppose $f_x(0,0) = f_y(0,0) = 0$ and $f_{xx}(0,0) = f_{yy}(0,0) = 2, f_{xy}(0,0) = 1$. Prove that there exists $\epsilon > 0$ such that $f(x, y) > f(0,0)$ for all $0 < x^2 + y^2 < \epsilon^2$ (so $f(x, y)$ has a local minimum at $(0,0)$).

4. [20%] Let $f(x)$ be a continuous function on the closed interval $[-1, 1]$. Show that
$$\lim_{n \to \infty} \int_{-1}^{1} f(x) \sin nx \, \mathrm{d}x = 0.$$

5. [20%] Let T be the surface in $\mathbb{R}^3$ given by the equation
$$(\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1$$
and $\vec{n}$ the outward unit normal vector. Evaluate the flux integral $\int_T \vec{F} \cdot \vec{n} \, \mathrm{d}S$ of the vector field
$$\vec{F}(x, y, z) = (0, 0, |z|).$$
(Here $\mathrm{d}S$ denotes the element of surface.)

試題隨卷繳回