- 本科目不得使用計算機

※請將選擇題作答於試卷內之「選擇題作答區」。

PART I: Multiple Choice (A), (B), (C), or (D). You do not need to justify your answer. (40%; 4% each.)

1. (A) $2^{3^2} = 8^2 = 64$.

   (B) $\tan^{-1}(\tan\pi) = \pi$.

   (C) If $f(x) = f^{-1}(x)$ for all $x \in \mathbb{R}$, then $f(x) = x$.

   (D) $\left(1+\frac{1}{n}\right)^n < 3$ for all $n \in \mathbb{N}$.

2. (A) If $f(x)$ is continuous on (a,b), then there is $c \in (a,b)$ such that $f(c) = \max\limits_{(a,b)} f(x)$.

   (B) Suppose that $f(x)$ is continuous but not differentiable at $x = 0$. The function $g(x) = xf(x)$ must be differentiable at $x = 0$.

   (C) If $f(x)$ is a differentiable function, then $\lim\limits_{x\to a} f'(x) = f'(a)$.

   (D) If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then there is a unique $c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$.

3. Consider the function $f(x) = \dfrac{1+\mathrm{e}^{-x^2}}{1-\mathrm{e}^{-x^2}}$.

   (A) $f(x)$ does not have any horizontal asymptotes and any vertical asymptotes.

   (B) $f(x)$ has a horizontal asymptote, but $f(x)$ does not have any vertical asymptotes.

   (C) $f(x)$ has a vertical asymptote, but $f(x)$ does not have any horizontal asymptotes.

   (D) $f(x)$ has a horizontal asymptote and a vertical asymptote.

4. (A) $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx \,\mathrm{d}x = 0$ for all $m, n \in \mathbb{N}$.

   (B) $\displaystyle\int_{-\pi}^{\pi} \sin mx \cos nx \,\mathrm{d}x = 0$ for all $m, n \in \mathbb{N}$.

   (C) $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx \,\mathrm{d}x = 0$ for all $m, n \in \mathbb{N}$.

   (D) $\displaystyle\int_{-\pi}^{\pi} \sin^m x \cos^n x \,\mathrm{d}x = 0$ for all $m, n \in \mathbb{N}$.

5. Consider the improper integral $I_p = \displaystyle\int_0^{\infty} \frac{1}{x^p}\,\mathrm{d}x$, where p is a positive number.

   (A) I_p is convergent if $p > 1$ and divergent if $0 < p \le 1$.

   (B) I_p is convergent if $0 < p < 1$ and divergent if $p \ge 1$.

   (C) I_p is convergent for all $p > 0$.

   (D) I_p is divergent for all $p > 0$.

6. The enclosed area of the curve $(x^2+y^2)^2 = x^2 - y^2$ can be written in the integral form as

   (A) $2\displaystyle\int_0^{\frac{\pi}{4}} \cos 2\theta \,\mathrm{d}\theta$.　　(B) $4\displaystyle\int_0^{\frac{\pi}{4}} \cos 2\theta \,\mathrm{d}\theta$.

   (C) $2\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{\cos 2\theta} \,\mathrm{d}\theta$.　　(D) $\dfrac{1}{2}\displaystyle\int_0^{\frac{\pi}{4}} \cos 2\theta \,\mathrm{d}\theta$.

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7. (A) If both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent, then $\sum_{n=1}^{\infty} a_n b_n = \left(\sum_{n=1}^{\infty} a_n\right) \cdot \left(\sum_{n=1}^{\infty} b_n\right)$.

   (B) If $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} (a_n + b_n)$ must be divergent.

   (C) Suppose that $f(x)$ is a positive and continuous function on $[1, \infty)$, and the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. Let $a_n = f(n)$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

   (D) If $a_n \le b_n$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ must be convergent.

8. (A) $\sum_{n=1}^{\infty} (-1)^n \left(1 - \cos \frac{1}{n}\right)$ is absolutely convergent.

   (B) $\sum_{n=1}^{\infty} \frac{1}{n - \ln n}$ is convergent.

   (C) $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ is convergent.

   (D) $\sum_{n=2}^{\infty} \frac{1}{n} \ln\left(\frac{n+1}{n-1}\right)$ is divergent.

9. Consider the function $f(x, y) = \sqrt{|xy|}$.

   (A) $f(x, y)$ is not continuous at $(0, 0)$.

   (B) $f(x, y)$ is continuous at $(0, 0)$, but partial derivatives $f_x(0, 0)$ and $f_y(0, 0)$ do not exist.

   (C) Both partial derivatives $f_x(0, 0)$ and $f_y(0, 0)$ exist, but $f(x, y)$ is not differentiable at $(0, 0)$.

   (D) $f(x, y)$ is differentiable at $(0, 0)$.

10. Consider the spherical coordinates system $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho \cos\phi$. The volume element $dV = dx\,dy\,dz$ can be changed as

   (A) $\rho \sin\phi \,d\rho\, d\theta\, d\phi$.

   (B) $\rho \cos\phi \,d\rho\, d\theta\, d\phi$.

   (C) $\rho^2 \sin\phi \,d\rho\, d\theta\, d\phi$.

   (D) $\rho^2 \cos\phi \,d\rho\, d\theta\, d\phi$.

PART II: Answer the following questions. (30%; 5% each.)

11. Suppose that a, b and c are constants and $\lim_{x \to -\infty} \left(\sqrt{ax^2 + bx + c} + 3x\right) = 2$, then $(a, b) =$ __(11)__.

12. Let $f(x) = e^x(e^x - 1)(e^x - 2) \cdots (e^x - 106)$. Find $f'(0) =$ __(12)__.

13. Suppose that $f(t)$ is a continuous function on $(1, \infty)$ satisfying $\int_2^{1+x^2} f(t)\,dt = \ln x$, then $f(10) =$ __(13)__.

14. Evaluate the definite integral $\int_{\frac{1}{2}}^{1} \frac{\sin^{-1} x}{x^2}\,dx =$ __(14)__.

15. The length of the curve $C : (x(t), y(t)) = \left(t^3 + 1, \frac{3}{2}t^2 - 1\right), 0 \le t \le 1$ is __(15)__.

16. Reverse the order of the iterated integral $\int_0^{\pi} \int_{\sin x}^{1} f(x, y)\,dy\,dx$: __(16)__.

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PART III: Solve the following problems. You need to write down complete arguments. (30%; 10% each.)

17. Given a family of parabolas $P_n : y = nx^2 + \frac{1}{n}$, where $n \in \mathbb{N}$. Let A_n be the area between P_n and P_{n+1}. Find the limit $\lim_{n\to\infty} \frac{A_n}{n^3}$.

18. Use the Taylor series method to find the limit $\lim_{x\to 0} \frac{\sqrt{1-x^4} - e^{2x^4}}{(1-\cos x)\sin^2 x}$.

19. Suppose that $f(t)$ is a continuous function on $[0, \infty)$ satisfying

$$f(t) = e^{\pi t^2} + \iint_{x^2+y^2 \le t^2} f\left(\sqrt{x^2+y^2}\right) \mathrm{d}A.$$

Find $f(t)$.

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