※注意：請於試卷上「非選擇題作答區」標明題號並依序作答。

Notation: $\mathbf{R}$ is the set of real numbers. $\mathrm{M}_n(\mathbf{R})$ is the set of $n \times n$ matrices with entires in $\mathbf{R}$ and $\mathbf{R}^n$ is n-dimensional column vectors over $\mathbf{R}$.

Problem 1 (20 pts). Let $T : \mathbf{R}^4 \to \mathbf{R}^3$ be the linear transformation defined by $T(v) = A \cdot v$, where

$$A = \begin{pmatrix} 1 & -1 & 0 & 3 \\ -1 & 2 & 1 & -1 \\ -1 & 1 & 0 & -3 \end{pmatrix} \in \mathrm{M}_{3\times 4}(\mathbf{R}).$$

(1) Find the rank and the nullity of T.
(2) Find bases of $\operatorname{Ker} T$ and $\operatorname{Im} T$.

Problem 2 (20 pts). Let

$$A = \begin{pmatrix} -1 & 4 & -2 \\ -2 & 5 & -2 \\ -1 & 2 & 0 \end{pmatrix}.$$

(1) Find the eigenvalues of A.
(2) Find an invertible matrix $P \in \mathrm{M}_3(\mathbf{R})$ such that $P^{-1}AP$ is a diagonal matrix.

Problem 3 (20 pts). Let $V = \mathrm{M}_3(\mathbf{R})$ be the 3-dimensional vector space over $\mathbf{R}$ given by

$$V = \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} \mid x_1, x_2, x_3 \in \mathbf{R} \right\}.$$

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and define the linear transformation $T : V \to V$ by

$$T(B) = ABA^{-1}.$$

(1) Write down the matrix representation A of T under the basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

(2) Find an invertible $P \in \mathrm{M}_3(\mathbf{R})$ such that $P^{-1}AP$ is the Jordan canonical form.

Problem 4 (20pts). Let $A \in \mathrm{M}_n(\mathbf{R})$ such that $A^n = 0$ but $A^{n-1} \neq 0$.
(1) Show that there exists $v \in \mathbf{R}^n$ such that $\{v, Av, A^2v, \ldots, A^{n-1}v\}$ is a basis of $\mathbf{R}^n$.
(2) If $B \in \mathrm{M}_n(\mathbf{R})$ such that $AB = BA$, prove that

$$B = a_0 + a_1 A + a_2 A^2 + \cdots a_{n-1} A^{n-1}$$

for some $a_0, \ldots, a_{n-1} \in \mathbf{R}$.

Problem 5 (20 pts). Let $A \in \mathrm{M}_n(\mathbf{R})$ be the matrix

$$A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{pmatrix}.$$

Show that for any non-zero $x \in \mathbf{R}^n$, $x^t A x > 0$.

試題隨卷繳回