1. Let A and B be two $n \times n$ matrices over a field F.
   (a) **(5 points.)** Show that $\operatorname{rank}(AB) \le \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$.
   (b) **(5 points.)** Show that $\operatorname{rank}(A+B) \le \operatorname{rank}(A) + \operatorname{rank}(B)$.
   (c) **(5 points.)** Show that $\operatorname{rank}(A) + \operatorname{rank}(B) \le \operatorname{rank}(AB) + n$.

2. **(15 points.)** Let $T : V \to V$ be a linear operator on a vector space V such that $\ker T = \ker T^2$ and $\operatorname{Im} T = \operatorname{Im} T^2$. (Here $\ker T$ and $\operatorname{Im} T$ denote the kernel and the image of T, respectively.) Prove that $V = \ker T \oplus \operatorname{Im} T$.

3. **(15 points.)** Let A be an $n \times n$ real matrix. Suppose that A is orthogonal, symmetric, and positive definite. Prove that A is the identity matrix.

4. **(15 points.)** Let $n \ge 2$ be an integer. For $i = 0, \ldots, n-1$, let $c_i = \binom{n}{i}$ denote the binomial coefficient. Let

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & 0 & \cdot & 0 & -c_1 \\ 0 & 1 & 0 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -c_{n-1} \end{pmatrix}.$$

   Determine the Jordan canonical form for A.

5. **(20 points.)** Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $Tv \perp v$ for all $v \in \mathbb{R}^3$ (with respect to the standard inner product on $\mathbb{R}^3$). Prove that T is not invertible.
   Is the analogous statement true for a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$?

6. Let S and T be two linear operators on a finite-dimensional vector space V over $\mathbb{C}$. Assume that S and T commute.
   (a) **(5 points.)** Let E_λ be an eigenspace of S with eigenvalue λ. Prove that E_λ is T-invariant.
   (b) **(15 points.)** Prove that there is a basis $\mathcal{B}$ for V such that $[S]_{\mathcal{B}}$ and $[T]_{\mathcal{B}}$ are both upper-triangular. (Here $[S]_{\mathcal{B}}$ denotes the matrix of S with respect to the basis $\mathcal{B}$.)

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