國立臺灣大學 114 學年度碩士班招生考試試題

※ 注意：請於試卷內之「非選擇題作答區」標明題號依序作答。

1. [6%] Let $\gamma(t)$ be a geodesic on a regular surface S. $p = \gamma(0)$. Suppose $\mathbf{V} \in S_p$ is an unit tangent vector which is perpendicular to $\gamma'(0)$. Describe the parallel transport of $\mathbf{V}$ along γ.
2. [6%] Let $S_1 = \{(x, y, 0) \,|\, x^2 + y^2 \geq 4\}$ and $S_2 = \{(\cos\theta, \sin\theta, z) \,|\, \theta \in \mathbb{R}, z \geq 1\}$. Let γ be the graph of the function $z = f(x)$ such that $1 \leq x \leq 2$, $f(1) = 1$, and $f(2) = 0$. Define Ω as the surface of revolution generated by rotating γ about the z-axis. Suppose the union of S_1, Ω, and S_2 forms a regular surface. What is the value of $\iint_\Omega K dA$?
3. [30%]Suppose $\gamma(s)$ is a regular closed space curve parametrized by arc length s. Let the surface S be defined by the parametrization $X(\theta, s) = \gamma(s) + \cos\theta\, \mathbf{n}(s) + \sin\theta\, \mathbf{b}(s)$, where $\mathbf{n}(s)$ and $\mathbf{b}(s)$ are the normal and subnormal vectors of $\gamma(s)$. Assume the curvature $\kappa(s)$ of γ satisfies $0 < \kappa(s) < 1$.
   a. [6%] Show that S is a regular surface.
   b. [6%] Compute $H(\theta, s)$ and $K(\theta, s)$.
   c. [6%] Express the principal directions at every point of S using X_θ and X_s. Is there any umbilical point on S?
   d. [6%] Define $\alpha_{s_0}(\theta)$ to be the curve $X(\theta, s_0)$. Is $\alpha_{s_0}(\theta)$ a line of curvature? Is $\alpha_{s_0}(\theta)$ a geodesic?
   e. [6%] What is the Euler characteristic $\chi(S)$ of S?
4. [18%] Let p be a point on a surface S, and let $\mathbf{V} \in S_p$ be a tangent vector. Let the curve $\gamma(s) \subset S$ be parametrized by arc length s, such that $\gamma(0) = p$ and $\gamma'(0) = \mathbf{V}$. Define $\tau_g(\mathbf{V}) = < \mathbf{N}'(0), \mathbf{N}(0) \times \mathbf{V}_p >$, where $\mathbf{N}(s)$ is the unit normal vector of S along $\gamma(s)$.
   a. [6%] Show that $\tau_g(\mathbf{V})$ is well-defined, i.e. it is independent of the choice of the curve $\gamma(s)$.
   b. [6%] If $\alpha(s)$ is a geodesic such that $\alpha(0) = p$ and $\alpha'(0) = \mathbf{V}$, show that $\tau_g(\mathbf{V}) = \tau(0)$, where $\tau(0)$ is the torsion of α at p.
   c. [6%] Suppose $\mathbf{V} = \cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2$, where $\mathbf{e}_i$ $(i = 1, 2)$ are the principal directions at p, satisfying $|\mathbf{e}_i| = 1$ and $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{N}_p$. Let κ_i $(i = 1, 2)$ denote the corresponding principal curvatures. Prove that
   $$\tau_g(\mathbf{V}) = (\kappa_1 - \kappa_2)\cos\theta\sin\theta.$$
5. [20%] Suppose S is a regular surface with non-zero mean curvature $H \neq 0$. It is stated that "the area of any compact domain contained in S decreases when the surface is deformed in the direction of $H\mathbf{N}$," where $\mathbf{N}$ is the unit normal vector of S. Clarify the meaning of this statement and provide a proof.
6. [20%] If S is a regular surface without umbilical points and has constant Gaussian curvature $K = 0$, prove that S is (locally) a ruled surface.

**** You might need Mainardi-Codazzi equations:

$$\begin{cases} e_v - f_u &= e\Gamma_{12}^1 + f(\Gamma_{12}^2 - \Gamma_{11}^1) - g\Gamma_{11}^2 \\ f_v - g_u &= e\Gamma_{22}^1 + f(\Gamma_{22}^2 - \Gamma_{21}^1) - g\Gamma_{21}^2 \end{cases}$$

試題隨卷繳回