1. (10 points) Let $g : [a, b] \to \mathbb{R}$ and $h : [a, b] \to \mathbb{R}$ be two continuous functions with $g(y) < h(y)$ for all $y \in (a, b)$. Consider

$$\Omega = \{(x, y) \in \mathbb{R}^2 : g(y) < x < h(y),\ a < y < b\}.$$

Prove (a special case of) Green's theorem: If $P : \Omega \cup \partial\Omega \to \mathbb{R}$ is continuous (where $\partial\Omega$ is the boundary of Ω), and if $\frac{\partial P}{\partial x} : \Omega \to \mathbb{R}$ is bounded and continuous, then

$$\iint_\Omega \frac{\partial P}{\partial x}(x, y)\ \mathrm{d}x\ \mathrm{d}y = \int_{\partial\Omega} P(x, y)\ \mathrm{d}y,$$

where the right-hand side is computed with respect to a counterclockwise parametrization of $\partial\Omega$.

2. (10 points) Let $f, g : [a, b] \to \mathbb{R}$ be two continuous functions such that they are differentiable on (a, b). Suppose that $|f'(x)| \ge |g'(x)| > 0$ for all $x \in (a, b)$. Prove that

$$|f(x) - f(y)| \ge |g(x) - g(y)| \text{ for all } x, y \in [a, b].$$

3. (15 points) Define

$$\operatorname{sign}(y) = \begin{cases} 1 & \text{if } y > 0, \\ 0 & \text{if } y = 0, \\ -1 & \text{if } y < 0, \end{cases} \qquad f(x) = \begin{cases} \operatorname{sign}\left(\sin\left(\frac{\pi}{x}\right)\right) & \text{if } x \in (0, 1], \\ 0 & \text{if } x = 0. \end{cases}$$

Prove or disprove: f is Riemann integrable.

4. (15 points) Let $\sum_{n=1}^{\infty} x_n$ be a divergent series, where each x_n is positive. Prove that there exists a divergent series $\sum_{n=1}^{\infty} y_n$ such that $y_n > 0$ for all n, and that

$$\lim_{n\to\infty} \frac{y_n}{x_n} = 0.$$

In other words, there is no "slowest" divergent series.

5. (12 + 8 points) We define a sequence of functions (f_n) as follows. Let $f_1(x) = x$. Once $f_n : [0, 1] \to \mathbb{R}$ is defined, we define $f_{n+1} : [0, 1] \to \mathbb{R}$ as

$$f_{n+1}(x) = f_n(x) + \frac{x - f_n(x)^2}{2}, \text{ for all } x \in [0, 1].$$

   (a) Prove that (f_n) converges pointwise, and determine the limiting function.

   (b) Does (f_n) converges uniformly on $[0, 1]$? Explain your answer.

6. (10 + 6 + 8 + 6 points) For $n \ge 1$ and $S \subset \mathbb{R}^n$ uncountable, define $\mathcal{C}(S)$ to be the set of all $x \in \mathbb{R}^n$ such that for each $\varepsilon > 0$, $B_x(\varepsilon)$ contains uncountably many points of S, where $B_x(\varepsilon) := \{y \in \mathbb{R}^n : d(x, y) < \varepsilon\}$ and d is the Euclidean distance.

   (a) Determine $\mathcal{C}(S)$ if $n = 1$ and if S is the middle-third Cantor set.

   (b) From now on, let $n \ge 1$ and $S \subset \mathbb{R}^n$ be uncountable.

      i. Show that $\mathcal{C}(S)$ is a closed set.

      ii. Show that $S \cap \mathcal{C}(S)^c$ is countable and $S \cap \mathcal{C}(S)$ is uncountable. (Here, $\mathcal{C}(S)^c$ is the complement of $\mathcal{C}(S)$ in $\mathbb{R}^n$.) You may use the following fact without giving a proof: for any $A \subset \mathbb{R}^n$, every open cover for A has a countable subcover.

      iii. For $E \subset \mathbb{R}^n$ and $x \in E$, we say that x is isolated if there exists $\delta > 0$ such that $B_\delta(x) \cap E = \{x\}$. Deduce from ii. that $\mathcal{C}(S)$ has no isolated points.