國立臺灣大學 114 學年度碩士班招生考試試題

(1) Let G be a finite abelian group of order m.
   (a) (10 %) Prove that for any positive integer $d|m$, there is a subgroup of order d.
   (b) (10 %) Give an example of a finite non-abelian group so that the above property does not hold.

(2) (10 %) Classify groups of order 21.

(3) The goal of this problem to solve for $x^5 - 1 = 0$ by radicals.
   (a) (15 %) Let $\zeta := e^{\frac{2\pi i}{5}}$ in $\mathbb{C}$. Show that $[\mathbb{Q}[\zeta] : \mathbb{Q}] = 4$ and verify your answer.
   (b) (10 %) Let $u := \zeta + \zeta^{-1}$. Show that $[\mathbb{Q}[u] : \mathbb{Q}] = 2$ and determine the minimal polynomial of ζ over $\mathbb{Q}[u]$.

(4) (15 %) Find all elements $x \in \mathbb{Z}/2025\mathbb{Z}$ so that $x^5 = 1$.

(5) In this problem, R denotes a commutative ring with identity with the extra properties that every ideal of R is finitely generated.
   (a) (10 %) $\mathfrak{N} := \{x \in R | x^n = 0 \text{ for some } n\}$. Verify that $\mathfrak{N}$ is an ideal of R. Prove that here is an integer $m > 0$ so that $\mathfrak{N}^m = 0$.
   (b) (10 %) Let $f : R \to R$ be a ring homomorphism. Show that if R is surjective, then R is an isomorphism.

(6) (10 %) Let K be an finite extension over F and D is an integral domain in between, that is, $F \subset D \subset K$. Prove that D is a field.

試題隨卷繳回