國立臺灣大學 114 學年度碩士班招生考試試題

※ 注意：請用 2B 鉛筆作答於答案卡，並先詳閱答案卡上之「畫記說明」。

1. (2%) Which of the following best defines a "singular solution" of a differential equation?
   A) A solution that satisfies the initial condition for all values of the independent variable.
   B) A solution that is independent of any particular initial condition.
   C) A solution that does not belong to the general solution family but still satisfies the differential equation.
   D) A unique solution that only exists for non-linear differential equations.

2. (2%) What does a "boundary-value problem" refer to in the context of differential equations?
   A) A differential equation problem that includes conditions only at a single point.
   B) A differential equation problem with conditions specified at multiple points in the domain.
   C) A type of problem that requires the use of Laplace transforms for its solution.
   D) A system of equations where the boundary is ill-defined.

3. (2%) Which of the following is the definition of a "homogeneous differential equation"?
   A) A differential equation where all terms are equal to a constant.
   B) A differential equation where the sum of derivatives equals zero.
   C) A differential equation in which the dependent variable and its derivatives are proportional.
   D) A differential equation in which all terms involve the dependent variable or its derivatives.

4. (2%) What is the "Wronskian" used to determine?
   A) Whether a given solution satisfies the differential equation.
   B) The order of a system of linear differential equations.
   C) The linear independence of solutions to a differential equation.
   D) The stability of a solution near an equilibrium point.

5. (2%) In solving systems of linear differential equations, what role do eigenvalues play?
   A) They define the growth or decay rates of the system's solutions.
   B) They provide the initial conditions required to solve the system.
   C) They specify the numerical method necessary to approximate solutions.
   D) They determine the time at which the system reaches equilibrium.

Note 1: For the following problems, assume the terms C, C_1, C_2, and C_3 are arbitrary constants.

Note 2: Some problems may use the following notation: $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2}$ and $y''' = \frac{d^3y}{dx^3}$

6. (4%) Find the solution to the differential equation:

$$\frac{dy}{dx} = \frac{2x}{y}, \qquad y(1) = 2$$

   A) $y = 2e^{x^2-1}$
   B) $y = \sqrt{2x^2}$
   C) $y = \pm\sqrt{2x^2}$
   D) $y = \sqrt{2x^2 + 2}$

7. (4%) Find the general solution to the differential equation:

$$\frac{dy}{dx} + y = e^{-x}$$

A) $y = \frac{1}{x+C}$

B) $y = e^{-x}x$

C) $y = e^{-x^2}(x + C)$

D) No Solution

8. (4%) Find the general solution to the differential equation:

$$\frac{dy}{dx} = \frac{3y - 6}{2x + 4}$$

A) $y = 2 + (x + 2)^{\frac{3}{2}}$

B) $y = 2 + C(x + 2)^{\frac{3}{2}}$

C) $y = 2 \pm C(x + 2)^{\frac{3}{2}}$

D) $y = 2 + (x + 2)^{\frac{3}{2}} + C$

9. (4%) Find the solution to the differential equation, given the constraints:

$$x\frac{dy}{dx} + y = x^3, \;\; x > 0$$

A) $y = \frac{x^3}{4}$

B) $y = \frac{x^2}{2} + C$

C) $y = \frac{x^3}{4} + \frac{C}{x}$

D) $y = \frac{x^3}{4} + \frac{C}{x^2}$

10. (4%) Find the general solution to the differential equation:

$$y''' - 6y'' - 12y' - 8y = 0$$

A) $y = C_1e^{2x}$

B) $y = C_1e^{2x} + C_2xe^{2x} + C_3e^{4x}$

C) $y = C_1e^{2x} + C_2xe^{2x} + C_3x^2e^{2x}$

D) $y = C_1e^{2x} + C_2x^2e^{2x} + C_3x^3e^{2x}$

11. (4%) Find the Laplace transform of the function:

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$$y(x) = x^2 e^{3x}$$

A) $Y(s) = \frac{2}{s^3}$

B) $Y(s) = \frac{2}{s^2-3}$

C) $Y(s) = \frac{2}{(s-3)^2}$

D) $Y(s) = \frac{2}{(s-3)^3}$

12. (4%) Find the inverse Laplace transform of the function:

$$Y(s) = \frac{s+5}{s^2+4s+5}$$

A) $y(x) = \cos(x) + 3\sin(x)$
B) $y(x) = e^{-2x}(\cos(x) + 3\sin(x))$
C) $y(x) = e^{-2x}(3\cos(x) + \sin(x))$
D) $y(x) = e^{-2x}\left(\cos(\sqrt{2}x) + \sqrt{2}\sin(\sqrt{2}x)\right)$

13. (4%) For what range of r does the following series converge?

$$\sum_{n=0}^{\infty} \frac{r^n}{n!}$$

A) All real values of r
B) $-1 < r < 1$
C) $-1 \le r \le 1$
D) No real values of r

14. (4%) Find the eigenvalues for the system of differential equations:

$$\frac{dx}{dt} = 4x + y, \qquad \frac{dy}{dt} = -2x + y$$

A) $\lambda_1 = -2,\ \lambda_2 = 1$
B) $\lambda_1 = -2,\ \lambda_2 = 4$
C) $\lambda_1 = -1,\ \lambda_2 = 4$
D) $\lambda_1 = 2,\ \lambda_2 = 3$

15. (4%) Solve the system of differential equations:

$$\frac{dx}{dt} = 3x - y, \qquad \frac{dy}{dt} = x + y$$

With initial conditions: $x(0) = 1,\ y(0) = -1$
A) $x(t) = e^{2t},\ y(t) = e^{2t}$
B) $x(t) = e^{2t},\ y(t) = te^{2t}$
C) $x(t) = e^{2t} - 2te^{2t},\ y(t) = e^{2t}$

D) $x(t) = e^{2t}(\cos(t) + \sin(t)),\ y(t) = e^{2t}(\cos(t) + \sin(t))$

16. (10%) For the system of linear equations $Ax = b$
   A) there may be no solutions
   B) there always is at most one solution
   C) there is always a solution
   D) there always are infinitely many solutions
   E) none of the preceding statements is true

17. (10%) Let A be a 4×7 matrix, and let B be an 7×4 matrix. Then
   A) $(AB)^T = B^T A^T$
   B) $AB^T = BA^T$
   C) $(AB)^T = A^T B^T$
   D) A^T is an 4×7 matrix
   E) none of the preceding statements is true

18. (10%) Suppose that A is a 5×5 matrix, and $k \neq 0$. Then:
   A) $\det(kA) = k \det A$
   B) $\det(kA) = k^5 \det A$
   C) $\det(kA) = 5k + \det A$
   D) $\det(kA) = k + \det A$
   E) none of the preceding statements is true

19. (10%) Let A be an arbitrary $n \times n$ matrix. Then
   A) The row space of A equals the null space of A
   B) The row space of A is contained in the column space
   C) The row space of A equals the column space of A
   D) The row space of A has the same dimension as the column space of A
   E) None of the preceding statements is true

20. (10%) Determine which statement is true for all $n \times n$ matrices A
   A) If v is an eigenvector of A, then v is an eigenvector of $P^{-1}AP$ for every invertible matrix P
   B) If λ is an eigenvalue of A, then is an eigenvalue of $P^{-1}AP$ for every invertible matrix P
   C) The diagonal entries of A are its eigenvalues
   D) None of the preceding statements are true
   E) If $Av = \lambda v$ for some vector v in R^n, then λ is an eigenvalue of A