國立臺灣大學 114 學年度碩士班招生考試試題

1. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed (i.i.d.) from normal distribution $N(0, \sigma^2)$. Consider the following estimators:

$$T_1 = \frac{1}{2}|X_1 - X_2| \text{ and } T_2 = \sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2}.$$

   (a) (5 points) Is T_1 unbiased for σ? Evaluate the mean square error (MSE) of T_1.

   (b) (6 points) Is T_2 unbiased for σ? If not, find a constant c such that cT_2 is unbiased for σ.

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with probability density function

$$f(x|\theta) = \frac{3x^2}{\theta^3}, \ 0 < x \le \theta;$$

   where $\theta > 0$.

   (a) (5 points) Find the maximum likelihood estimator of θ.

   (b) (5 points) Find the method of moments estimator of θ.

   (c) (7 points) Obtain a pivotal quantity of θ. A pivotal quantity is defined as a function of the data and the parameter θ but whose distribution is free of the parameter.

   (d) (5 points) Use the pivotal quantity from (c) to derive a $(1-\alpha) \times 100\%$ two-sided confidence interval for θ.

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with probability density function

$$f(x|\theta) = \frac{2x}{\theta} e^{-x^2/\theta}, \ x > 0;$$

   and $f(x|\theta) = 0$ for $x \le 0$.

   (a) (5 points) Show that X_1^2 is an unbiased estimator of θ.

   (b) (7 points) Find the Cramér–Rao Lower bound (CRLB) for the variance of an unbiased estimator of θ.

   (c) (5 points) Find the uniformly minimum variance unbiased estimator (UMVUE) of θ.

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4. Let $X_1, X_2, \ldots, X_n$ denote a series of random variables sampled from a distribution with probability density function $f(x;\theta)$. Suppose that a test of level α is required to evaluate $H_0 : \theta = \theta_0$ and $H_1 : \theta = \theta_1$.

   (a) (10 points) State and prove Neyman-Pearson lemma.

   (b) (5 points) Assume that $X_1, X_2, \ldots, X_n$ are sampled from $N(\mu, 1)$. Find a best critical region for $H_0 : \mu = 0$ and $H_1 : \mu = 1$ at level α.

5. Let X denote a sample form a distribution ξ. Suppose that a test of level α is required to evaluate $H_0 : \xi = \xi_0$ and $H_1 : \xi = \xi_1$, where ξ_i is a known distribution with probability density function $f_i(\cdot)$. For any $\alpha > 0$, define

$$T_\alpha(X) = \begin{cases} 1 & \text{if } f_1(X) > c(\alpha) f_0(X); \\ r(\alpha) & \text{if } f_1(X) = c(\alpha) f_0(X); \\ 0 & \text{if } f_1(X) < c(\alpha) f_0(X), \end{cases}$$

where $0 \le r(\alpha) \le 1$, $c(\alpha) \ge 0$, and $E[T_\alpha(X) | \xi = \xi_0] = \alpha$.

   (a) (10 points) Show that $c(\alpha_1) \ge c(\alpha_2)$ if $\alpha_1 < \alpha_2$.

   (b) (10 points) Show that the type II error probability of $T_{\alpha_1}(X)$ is larger than the type II error probability of $T_{\alpha_2}(X)$ if $\alpha_1 < \alpha_2$.

6. (15 points) Let $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ denote two independent multivariate normal random vectors given by

$$\boldsymbol{Y}_1 \sim N(\gamma_{10}\mathbf{1}_n + \boldsymbol{Z}\gamma_1, \sigma^2 \boldsymbol{I}_n) \text{ and } \boldsymbol{Y}_2 \sim N(\gamma_{20}\mathbf{1}_n + \boldsymbol{Z}\gamma_2, \sigma^2 \boldsymbol{I}_n),$$

where γ_{10} and γ_{20} are parameters, γ_1 and γ_2 are $q \times 1$ parameter vectors, and $\boldsymbol{Z}$ is an $n \times q$ full-column-rank matrix. Note also that $\mathbf{1}_n$ is the $n \times 1$ vector of ones and $\boldsymbol{I}_n$ is the identity matrix of order n. Find a likelihood ratio test of level α for $H_0 : \gamma_1 = \gamma_2$ and $H_1 : \gamma_1 \neq \gamma_2$.