(1)

(a) (10%) Find the eigenvalues and corresponding eigenvectors for the following matrix A.

$$A = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \vdots & \vdots \\ c_1 & c_2 & \cdots & c_n \end{bmatrix}$$

(b) (5%) Let

$$A = \begin{bmatrix} -4 & -6 & 0 \\ 3 & 5 & 0 \\ 3 & 3 & 2 \end{bmatrix}$$

Find the characteristic polynomial and eigenvalues of A.

(c) (5%) Continuing from (b), find, if possible, an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Otherwise, explain why A is not diagonalizable.

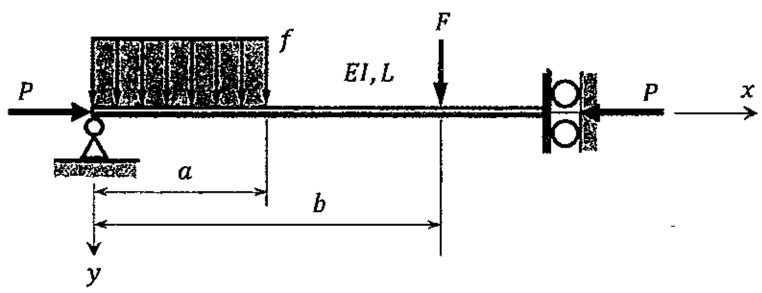
(2) Let $f(x,y) = e^{xy}\cos(x+y)$.

(a) (5%) In what direction, starting at $(0, \pi/2)$, is f changing the fastest?

(b) (4%) In what direction, starting at $(0, \pi/2)$, is f changing at 50% of its maximum rate?

(c) (5%) Let $\boldsymbol{c}(t)$ be a flow line of $\boldsymbol{F} = \nabla f$ with $\boldsymbol{c}(0) = (0, \pi/2)$. Calculate $\frac{d}{dt}[f(\boldsymbol{c}(t))]\Big|_{t=0}$

(3) Consider the beam-column as shown below, where $P = 1$ and $EI = 1$.



The differential equation can be expressed as

$$EI\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} + P\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = w(x)$$

where $w(x)$ is the external load.

Please answer the following questions:

(a) (6%). Use the Heaviside step function and the Dirac delta function to express the external load $w(x)$.

(b) (4%). Show the boundary conditions in terms of y.

(c) (10%). If $Y(s) = \mathcal{L}\{y(x)\}$ ($\mathcal{L}$ is the Laplace transform), please solve $Y(s)$.

(4) Consider the following equation:

$$2x^2 y'' + x(2x+1)y' - y = 0$$

(a) (3%). Is $x = 0$ a singular point or a regular singular point?

(b) (10%). If the equation has a Frobenius series solution of the form

$$y(x) = x^{\alpha}\sum_{n=0}^{\infty} a_n x^n, \quad a_0 \neq 0, \quad 0 < x < \infty$$

where $a_n, n = 0, 1, \ldots,$ are constants to be determined. Please solve all α that satisfies the solution.

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(5) Consider the following three partial differential equations:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0, \quad (5\text{-}1)$$

$$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=\frac{\partial u}{\partial t}, \quad (5\text{-}2)$$

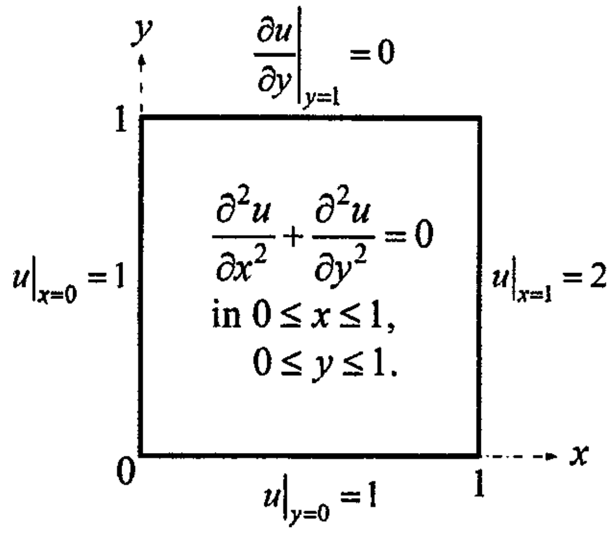
$$c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=\frac{\partial^2 u}{\partial t^2}, \quad (5\text{-}3)$$

where u is a physical property which is a function of the spatial coordinates (x, y, z) and time t, and k and c are some material properties, which are assumed to be constants.

(a) (7%) What are the names of the equations, and provide physical interpretation for each equation through a specific example.

(b) (6%) If those three equations are to be solved separately in a cubic box with unit edge length, specify the appropriate boundary conditions for solving the equations, together with initial conditions (if any).

(6) (20%) Solve the problem as shown in the figure below. The function $u\ (x, y)$ satisfies the Laplace equation within the domain $0 \le x \le 1,\ 0 \le y \le 1$, subjected to the boundary conditions as indicated next to the four boundaries at $x = 0$, $x = 1$, $y = 0$ and $y = 1$, respectively.



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