1. (a) **(5 points.)** Let G be a group and d be a divisor of $|G|$. Let X be the set of elements of order d in G. Prove that the function $* : (g, x) \mapsto gxg^{-1}$ defines a group action of G on X.
   (b) **(15 points.)** Assume that G is a group of order 56 with 8 different Sylow 7-subgroups. Prove that there is a unique Sylow 2-subgroup and that it is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.
2. (a) **(10 points.)** Show that $\mathbb{Z}[\sqrt{-2}] := \{a + b\sqrt{-2} : a, b \in \mathbb{Z}\}$ is a Euclidean domain.
   (b) **(10 points.)** Find the GCD of $32 + 52\sqrt{-2}$ and $24 + 36\sqrt{-2}$ in $\mathbb{Z}[\sqrt{-2}]$.
3. **(20 points.)** Let R be a commutative ring with 1 and $M(n, R)$ be the ring of $n \times n$ matrices over R. Show that every (two-sided) ideal of $M(n, R)$ is of the form $M(n, I)$ for some ideal I of R.
4. **(20 points.)** Let $a, b \in \mathbb{Z}$. Find the necessary and sufficient conditions such that $\mathbb{Q}(\sqrt{a + \sqrt{b}})$ is a cyclic extension of degree 4 over $\mathbb{Q}$. (A field extension E/F is said to be a cyclic extension if it is a Galois extension and the Galois group is cyclic.)
5. Let $f(x)$ be an irreducible polynomial of degree 4 over $\mathbb{Q}$ and $\alpha_1, \ldots, \alpha_4$ be the zeros of $f(x)$ in $\overline{\mathbb{Q}}$. Let

$$\beta_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4, \quad \beta_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4, \quad \beta_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3.$$

   (a) **(10 points.)** Prove that $g(x) = (x - \beta_1)(x - \beta_2)(x - \beta_3)$ is a polynomial in $\mathbb{Q}[x]$. That is, show that $\beta_1 + \beta_2 + \beta_3$, $\beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1$, and $\beta_1\beta_2\beta_3$ are all in $\mathbb{Q}$.
   (b) **(10 points.)** Let E be the splitting field of $f(x)$. Prove that $\mathrm{Gal}(E/\mathbb{Q})$ is isomorphic to S_4 or A_4 if and only if $g(x)$ is irreducible over $\mathbb{Q}$.