In this exam, without further specifying, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{C}$ denotes the set of all complex numbers, and i denotes the imaginary unit. For a complex number $a+b\mathrm{i}$ with $a,b\in\mathbb{R}$, we denote $(a+b\mathrm{i})^\dagger$ as its complex conjugate, i.e. $a-b\mathrm{i}$. We will also use '†' to denote the *conjugate transpose* of a complex matrix. We let I_n to be the $n\times n$ identity matrix. We will use $\mathbf{0}$ to denote the zero vector or the zero matrix in some vector space.

1. (Basis and Dimension) This problem concerns the basic definition of vector space, and its basis and dimension.

   (a) (5%) Let $\mathbb{Z}_2^n := \{0,1\}^n$ the set of all n-bit strings for any integer n. The set $\mathbb{Z}_2^n$ forms a vector space over $\mathbb{Z}_2$. For example, for vectors $101, 001 \in \mathbb{Z}_2^3$ and scalars $0,1\in\mathbb{Z}_2$, we have

$$101+001 \pmod 2 := (1\oplus 0)(0\oplus 0)(1\oplus 1) = 100 \in \mathbb{Z}_2^3;$$
$$101\cdot 1 = (1\cdot 1)(0\cdot 1)(1\cdot 1) = 101 \in \mathbb{Z}_2^3;$$
$$101\cdot 0 = (1\cdot 0)(0\cdot 0)(1\cdot 0) = 000 \in \mathbb{Z}_2^3.$$

   What is the dimension of the vector space $\mathbb{Z}_2^n$ over $\mathbb{Z}_2$? Give a basis of the vector space $\mathbb{Z}_2^n$ over $\mathbb{Z}_2$.

   (b) (5%) We denote '†' by a *conjugate transpose*. For example:

$$A = \begin{bmatrix} 1 & -2-\mathrm{i} & 5 \\ 1+\mathrm{i} & \mathrm{i} & 4-2\mathrm{i} \end{bmatrix} \in \mathbb{C}^{2\times 3};$$
$$A^\dagger = \begin{bmatrix} 1 & 1-\mathrm{i} \\ -2+\mathrm{i} & -\mathrm{i} \\ 5 & 4+2\mathrm{i} \end{bmatrix}.$$

   A (possibly complex) matrix A is Hermitian if and only if $A^\dagger = A$. Consider a vector space $\{A\in\mathbb{C}^{n\times n} : A^\dagger = A\}$ over field of real numbers. What is its dimension?

2. (Matrix Inversion)

   (a) (5%) Let A be an $n\times n$ non-singular matrix that satisfies $A^3-4A^2+3A-5I_n=\mathbf{0}$. Calculate the inverse of A in terms of a polynomial of A.

   (b) (5%) Let $\omega := \mathrm{e}^{\frac{2\pi\mathrm{i}}{n}}$ for some integer n and let

$$B := \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}.$$

   Calculate the inverse of B. Express your answer in the most simplified form.

   (c) (5%) Justify your answers to Problem (b).

3. (5%) Suppose columns of a matrix A are n vectors in $\mathbb{R}^m$. Answer the following questions.
   (a) (True or False) A is an $n \times m$ matrix.
   (b) If the columns are linear independent, what is the rank of A?
   (c) If the columns span $\mathbb{R}^m$, what is the rank of A?
   (d) If the columns form a basis for $\mathbb{R}^m$, what can you say about the rank, n, and m?
   (e) Suppose A has rank r, it means that A has r ______ columns?
   (f) (True or False) The map $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathrm{x}) := A\mathrm{x}$ is a linear transform.

   (Getting 5 points if all answers are correct. Otherwise, 0 point.)

4. (Eigenvalues and Eigenvectors)
   (a) (5%) Let
   $$A := \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}.$$
   Write down the 3 eigenvalues of A with multiplicity in the decreasing order.
   (b) (10%) Suppose that an $n \times n$ matrix B satisfies
   $$B := \begin{bmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{bmatrix},$$
   where $a > 0$ and $b > 0$. Write down all the eigenvalues of B with multiplicity in the decreasing order in terms of a, b, and n.
   (c) (5%) Write down all the eigenvectors associated with the above eigenvalues of B. Note that each eigenvector has to be normalized to have a unit Euclidean norm.

5. Consider a random variable X and $\mathrm{E}[|X|] < \infty$. Hence, its expectation $\mathrm{E}[X]$ exists. Let us denote $\mathrm{E}[X]$ as μ_X for notational simplicity. The absolute deviation from the mean is $|X - \mu_X|$, and its expectation is denoted as
   $$d_X := \mathrm{E}\big[|X - \mu_X|\big].$$
   Let σ_X denote the standard deviation of X if it exists.
   (a) (5%) Suppose the probability density function of X, $f_X(t)$, is proportional to $e^{-\lambda|t|}$, for some $\lambda > 0$. Derive d_X in terms of σ_X.
   (b) (5%) Let X be a normal random variable. Derive d_X in terms of σ_X.
   (c) (5%) Is it true that for any random variable X with finite variance, $d_X \leq \sigma_X$? If your answer is "yes", prove it. If your answer is "no", give a counter example.

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6. Let X be continuous random variable with cumulative distribution function $F_X(t), t \in \mathbb{R}$ and probability density function $f_X(t), t \in \mathbb{R}$. Furthermore, $f_X(t) = f_X(-t)$ for any $t \in \mathbb{R}$, and $\mathrm{E}[X^2] < \infty$. Let Y be another random variable, independent of X, that takes values at 1 or -1 with equal probability, that is,

$$Y = \begin{cases} 1, & \text{with probability } 1/2 \\ -1, & \text{with probability } 1/2 \end{cases}$$

Let $Z = XY$, the product of X and Y.

(a) (5%) Are X and Z correlated? Justify your answer rigorously by deriving the covariance between X and Z.

(b) (5%) Are X and Z independent? Justify your answer rigorously by deriving the joint cumulative distribution function of X and Z.

7. Let U be an uniform random variable over the interval $(0, 1)$. Given $U = u$, $X_1, X_2, \ldots$ are independent and identically distributed Bernoulli u random variables. Let W_n denote the number of "1"s in the length-n sequence $(X_1, X_2, \ldots, X_n)$.

(a) (5%) Derive the conditional probability mass function of $(X_1, X_2, \ldots, X_n, W_n)$ given U:

$$P_{X_1,X_2,\ldots,X_n,W_n|U}(x_1, x_2, \ldots, x_n, w|u).$$

(b) (5%) Derive the conditional probability mass function of $(X_1, X_2, \ldots, X_n)$ given W_n:

$$P_{X_1,X_2,\ldots,X_n|W_n}(x_1, x_2, \ldots, x_n|w).$$

(c) (5%) Derive the moment generating function of W_n.

(d) (5%) Derive the probability mass function of W_n.

(e) (5%) Derive the joint probability mass function of $(X_1, X_2, \ldots, X_n)$:

$$P_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n).$$

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