國立臺灣大學 111 學年度碩士班招生考試試題

(1) In this problem, we will consider some properties of finite groups
  (a) (10 pts) Classify groups of order 8 up to isomorphism.
  (b) (5 pts) Let S_4 be the permutation group on 4 elements and let G be the Sylow 2-subgroup of S_4. Determine the structure of G (e.g. identify G to be one of those on your list in problem (1)).
  (c) (10 pts) Show that there is a surjective group homomorphism from S_4 to S_3 and determine its kernal.

(2) Let R be a commutative ring with identity and $I, J \lhd R$ are two ideals.
  (a) (5 pts) Show that $I \cap J, I + J$ and IJ are ideals of R.
  (b) (5 pts) Suppose that $I + J = R$. Prove that $I \cap J = IJ$.
  (c) (5 pts) Give an example that $I \cap J = IJ$ but $I + J \neq R$.
  (d) (10 pts) Suppose that $I + J = R$. Prove that $R/IJ \cong R/I \times R/J$.

(3) Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial with coefficient in $\mathbb{Q}$.
  (a) (5 pts) Let $u \in \mathbb{C}$ be a root of $f(x)$. Prove that $\mathbb{Q}[u] = \mathbb{Q}(u)$.
  (b) (5 pts) Let u_1, u_2 be any two roots of $f(x)$. Prove that there is an isomorphism of fields $\phi : \mathbb{Q}(u_1) \xrightarrow{\cong} \mathbb{Q}(u_2)$.
  (c) (10 pts) Let K be the splitting field of $f(x)$ over $\mathbb{Q}$. Prove that there is an isomorphism of K extending ϕ.
  (d) (10 pts) Let $f(x) = x^3 + x + 2022$. Determine its Galois group.

(4) Let N be a free abelian group (for example, $\mathbb{Z} \oplus \mathbb{Z}$) with a basis $\{u, v\}$ and M be a free abelian group with a basis $\{x, y, z\}$. We consider a group homomorphism ϕ such that $\phi(u) = 18x + 60y + 18z$ and $\phi(v) = 24x + 60y + 12z$. Let $G = M/\mathrm{im}(\phi)$.
  (a) (5 pts) Determine the rank of G.
  (b) (5 pts) Determine the structure of the torsion part of G.
  (c) (10 pts) Find a subset S of M so that the image of S in G forms a basis of the free part of G.

試題隨卷繳回