Linear Algebra

1. **(20 points.)** Let $A, B \in M_{n\times n}(F)$ be two $n \times n$ matrices over a field F.
   (a) Prove that $\operatorname{rank}(A+B) \le \operatorname{rank}(A) + \operatorname{rank}(B)$.
   (b) Prove that $\operatorname{rank}(A) + \operatorname{rank}(B) \le \operatorname{rank}(AB) + n$.
2. **(15 points.)** Let A be an $n \times n$ matrix over $\mathbb{C}$ of the form

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_0 \end{pmatrix}.$$

   Define $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1}$ and $\omega = e^{2\pi i/n}$. Prove that

$$\det A = \prod_{j=0}^{n-1} f(\omega^j).$$

3. **(15 points.)** Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V over $\mathbb{C}$ and $f(x) \in \mathbb{C}[x]$ be a polynomial. Prove that the linear transformation $f(T)$ is invertible if and only if $f(x)$ and the minimal polynomial T have no common roots.
4. **(15 points.)** Let $v_1, \ldots, v_k$ be eigenvectors corresponding to k distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ of a linear operator T on a vector space V. Prove that the T-cyclic subspace generated by $v = v_1 + \cdots + v_k$ has dimension k.
5. **(15 points.)** Let $T : V \to V$ be a linear operator on a finite-dimensional inner product space V over $\mathbb{R}$ and T^* be its adjoint. Suppose that $T^* = T^3$. Prove that T^2 is diagonalizable over $\mathbb{R}$.
6. **(20 points.)** Let V be a vector space of dimension n over a field F. Determine the dimension over F of the vector space of multilinear alternating functions $f : V \times \cdots \times V \to F$ (k copies of V).