國立臺灣大學 110 學年度碩士班招生考試試題

※ 注意：請於試卷上「非選擇題作答區」內依序作答，並應註明作答之大題及其題號。

Any device with computer algebra system is prohibited during the exam. Solve the following problems. You need to write down your reasoning.

1. Suppose that $f(x)$ satisfies the equation $f(x+y) = f(x) + f(y) + xy - x^2y - xy^2$ for all $x, y \in R$ and $\lim_{x\to 0} \frac{f(x)}{x} = 2$.
   (a) (6 pts) Find $f(0)$ and $f'(x)$.
   (b) (6 pts) Sketch the graph of $f(x)$, indicating intervals of increasing/ decreasing, and concavity.

2. A man runs twice as fast as he swims. He is at point A on the edge of a circular pool with radius 20 meters and he wants to get to the diametrically opposite point B as quickly as possible. He can run around the edge to a point C and then swim directly from C to B.
   (a) (10 pts) How should he choose the point C to minimize the total time ?
   (b) (6 pts) If he runs m times as fast as he swims, how will his best strategy be modified as m varies ($m \geq 1$) ?

3. Investigate the integral $\int_0^1 \frac{\ln x}{1+x^2} dx$.
   (a) (7 pts) Show that the improper integral $\int_0^1 \frac{\ln x}{1+x^2} dx$ converges. Show that
   $$\int_0^1 \frac{\ln x}{1+x^2} dx = -\int_1^\infty \frac{\ln x}{1+x^2} dx.$$
   (b) (4 pts) Write $\frac{1}{1+x^2}$ as the sum of a power series $\sum a_n x^n$.
   (c) (6 pts) Write $\int_0^1 \frac{\ln x}{1+x^2} dx = \int_0^1 (\sum a_n x^n) \ln x \, dx$ as the sum of a series. Thus, we can use its partial sums to estimate the integral.

4. (10 pts) Find the twice differentiable function $f(x)$ such that
   $$f'(x) = \int_0^x \sqrt{1+(f'(u))^2} du, \quad f(0) = 2.$$

5. (a) (5 pts) Let $f(x, y, z) = \int_{x^z}^{\sqrt{y}} \sin(t^2) \, dt$. Find ∇f, the gradient of f.
   (b) (5 pts) $f(x, y) = \frac{\sin(xy^2)}{x^2+y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Compute the directional derivative of f along $\mathbf{u} = (\cos\theta, \sin\theta)$ at $(0, 0)$.

6. (10 pts) Find the critical points of $f(x, y)$, where $z = f(x, y)$ satisfies the equation $yz + x\ln y = z^2$. Are these critical points local maximum, local minimum, or saddle points?

7. Evaluate the following multiple integrals.

(a) (7 pts) $\int_0^{\frac{\sqrt{3}}{2}} \int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} e^{x^2+y^2}\, dxdy + \int_{\frac{\sqrt{3}}{2}}^{\sqrt{3}} \int_{\frac{y}{\sqrt{3}}}^{\sqrt{4-y^2}} e^{x^2+y^2}\, dxdy$

(b) (8 pts) $\iiint_E \frac{1}{1+z} dV$, where $E = \{(x,y,z) | x^2+y^2+z^2 \le 1, x \ge 0, y \ge 0, z \ge 0\}$.

8. (10 pts) Let S be the part of the cylinder $x^2+y^2=2y$ that lies in the sphere $x^2+y^2+z^2=4$ and inside the first quadrant. Compute $\iint_S z\, dS$.