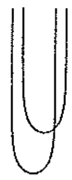
※ 注意：請於試卷內之「非選擇題作答區」標明題號依序作答。

(1) Let $\gamma : (-\frac{\pi}{2}, \frac{\pi}{2}) \mapsto \mathbf{R}^2$ be a plane curve defined by $\gamma(x) = (x, -\ln(\cos(x)))$.
(a) ([10%]) Find the curvature of γ and show that $(0,1)^\perp = \kappa n$ where κ is the curvature and n is the unit normal vector on γ and $V^\perp = (V \cdot n)n$.
(b) ([10%]) Let $\Gamma_T : (-\frac{\pi}{2}, \frac{\pi}{2}) \mapsto \mathbf{R}^2$ be a family of plane curves defined by

$$\Gamma_T(x) = (x + \frac{\pi}{4}, -\ln(\cos(x)) + T) = \gamma(x) + (\frac{\pi}{4}, T)$$

where $T \in \mathbf{R}$. Show that Γ_T intersects γ at exactly one point for each $T \in \mathbf{R}$ and the angle between the tangent vectors of γ and Γ_T at the intersection point is independent of T.

γ and Γ_0



γ and Γ_3

(2) ([25%]) Prove that if a regular surface in $\mathbf{R}^3$ contains a straight line, then the surface has non-positive Gauss curvature at all the points of this line.

(3) ([30%]) Consider a regular surface S where $0 \notin S$ and the inversion $\phi : \mathbf{R}^3 \setminus \{0\} \mapsto \mathbf{R}^3 \setminus \{0\}$ given by

$$\phi(p) = \frac{p}{||p||^2}, \text{ for } p \in \mathbf{R}^3 \setminus \{0\}.$$

Compare the mean curvature and the Gauss urvatures of S and of the image surface $S' = \phi(S)$. More precisely, show that $H'(\phi(p)) = ||p||^2 H(p) + 2\langle N(p), p\rangle$ and $K'(\phi(p)) = ||p||^4 K(p) + 4||p||^2 H(p) + 4\langle N(p), p\rangle^2$ where $N(p)$ is the unit normal of S at p, $H'(\phi(p))$ is the mean curvature of S' at $\phi(p)$, $K'(\phi(p))$ is the Gauss curvature of S' at $\phi(p)$.

(4) ([25%]) Prove that if S is a regular surface that is diffeomorphic to a cylinder and has Gaussian curvature $K < 0$, then S has at most one simple closed geodesic (up to reparametrization).

試題隨卷繳回