國立臺灣大學 110 學年度碩士班招生考試試題

※ 注意：請於試卷內之「非選擇題作答區」作答，並應註明作答之題號。

1. Classify groups of order n up to isomorphism in the following two cases.
   (a) [5%] $n = 2021 = 43 \times 47$.
   (b) [15%] $n = 2020 = 4 \times 5 \times 101$.

2. [20%] Let A be a commutative ring with identity, and $\mathfrak{m}$ a maximal ideal. Show the following statements are equivalent:
   (i) A has only one maximal ideal.
   (ii) $A \setminus \mathfrak{m}$ consists of units in A.
   (iii) If a, b are not unit, then $a + b$ is not a unit.

3. [20%] Prove the following simple form of the structure theorem for finitely generated modules over a principal ideal domain (so you can *not* apply the structure theorem directly). Let A be a principal ideal domain and M a 2×2 matrix whose entries are in A. Show that there exist invertible matrices P, Q with entries in A and $\alpha, \beta \in A$ with $\alpha \mid \beta$ such that $PMQ = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$.

4. (a) [5%] For a positive integer n, let $\phi(n)$ denote the cardinality of invertible elements in the ring $\mathbb{Z}/n\mathbb{Z}$. Show that
   $$\phi(n) = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right)$$
   where p runs through all primes dividing n.
   (b) [10%] Determine the cardinality of invertible 2×2 matrices with coefficients in $\mathbb{Z}/n\mathbb{Z}$ in terms of $\phi(n)$.
   (c) [5%] Determine the cardinality of invertible 2×2 matrices with coefficients in $\mathbb{Z}/n\mathbb{Z}$ whose determinants are equal to 1 in terms of $\phi(n)$.

5. Let n be an integer and $n \geq 2$. Following the steps below to show that $x^n - x - 1$ is irreducible over $\mathbb{Q}$. (You can provide another proof and get the credits.)

   For a polynomial $f(x) = a_n x^n + \cdots + a_k x^k + \cdots + a_0 \in \mathbb{Q}[x]$ of degree n, let $\tilde{f}(x) = x^n f(1/x) = a_0 x^n + \cdots a_k x^{n-k} + \cdots a_n$.

   (a) [4%] Let $f(x) \in \mathbb{Z}[x]$ be a monic with $f(0) = \pm 1$ such that $f(x), \tilde{f}(x)$ have no common root in $\mathbb{C}$. Suppose $f(x) = g(x)h(x)$ for non-constant $g(x), h(x) \in \mathbb{Z}[x]$. Show that there exists a monic $k(x) \in \mathbb{Z}[x]$ with $k \neq \pm f, \pm\tilde{f}$ such that $f\tilde{f} = k\tilde{k}$.
   (b) [8%] Show that for $n \geq 2$ and $f(x) = x^n - x - 1$, the two polynomials $f, \tilde{f}$ have no root in common in $\mathbb{C}$.
   (c) [8%] Let $f(x) = x^n - x - 1$ for $n \geq 2$. Show that if a monic $k(x) \in \mathbb{Z}[x]$ satisfies $f\tilde{f} = k\tilde{k}$, then $k = f$ or $-\tilde{f}$.

試題隨卷繳回