考生應作答於「答案卡」(請勿作答於試卷之選擇題作答區)，未作答於答案卡者不予計分。

In this exam, without specifying, all matrices are real-valued. $\mathbb{R}$ denotes the set of all real numbers, and $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathsf{M}_{m\times n}(\mathbb{R})$ and $\mathsf{M}_{m\times n}(\mathbb{C})$ denote the collections of all real-valued and complex-valued $m \times n$ matrices, respectively. For a matrix A, $A_{i,j}$ denotes its (i, j)-th entry. For a vector x, x_i denotes its i-th entry. I_n denotes the $n \times n$ identity matrix. j denotes $\sqrt{-1}$.

一、單選題：請作答於「答案卡」，依題號順序作答 (1., 2., ..., 10.)。

每題有三個選項。For each of the following statements, please determine whether it is (A) true, (B) false, or (C) uncertain (insufficient information to determine). If the statement is sometimes true and sometimes false, you should choose (C).

1. (5%) Let A and B be two $n \times n$ symmetric matrices, $n \geq 1$.

   Then, $\text{rank}(AB) = \text{rank}(BA)$.

2. (5%) Let A and B be two matrices such that the matrix product AB is well defined.

   Then, $\text{rank}(AB) > \text{rank}(A)$ and $\text{rank}(AB) > \text{rank}(B)$.

3. (5%) Let A be a $n \times m$ matrix, $n > m \geq 1$, and $\text{rank}(A) = m$.

   Then, there does not exist any $m \times n$ matrix B such that $BA = I_m$.

4. (5%) Let A be a $n \times n$ invertible matrix, $n \geq 1$. Let v be an eigenvector of A.

   Then, v is also an eigenvector of A^{-1}.

5. (5%) Let A, B be $n \times n$ matrices, $n \geq 1$, and λ be an eigenvalue of the matrix product AB.

   Then, λ is also an eigenvalue of the matrix product BA.

6. (5%) Let A be a $n \times n$ symmetric matrix, $n \geq 1$, and λ be an eigenvalue of A.

   Then, λ is a real number.

7. (5%) Let A be a $n \times n$ skew-symmetric matrix, $n \geq 1$, that is, $A^T = -A$.

   Then, $\det(A) \geq 0$.

8. (5%) Consider a system of linear equations $Ax = b$, where A is a 2×3 matrix, x is a 3×1 column vector, and b is a 2×1 column vector. Let $A_{i,j} \neq 0$, $\forall i = 1, 2,\ j = 1, 2, 3$. In other words, all entries of A are non-zero.

   Then, it is impossible to uniquely determine x_1 from the above system of linear equations.

9. (5%) Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be subspaces of $\mathbb{R}^n$ and $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$, the *sum* of the two subspaces. Let $\mathcal{D}_1 = \mathcal{S} \setminus \mathcal{S}_2$, the set of elements in $\mathcal{S}$ but not in $\mathcal{S}_2$.

   Then, $\mathcal{D}_1$ is also a subspace, and its dimension $\dim(\mathcal{D}_1) = \dim(\mathcal{S}_1)$.

10. (5%) Let $\mathcal{S}$ be a subspace of $\mathbb{R}^n$, $n > 1$. Let $\dim(\mathcal{S}) = m$, $1 < m < n$. Let A be a $n \times n$ diagonal matrix with positive diagonal entries, but not all entries are equal. Consider the following set:

$$\mathcal{T} = \{x \mid x^T A u = 0,\ \forall u \in \mathcal{S}\}.$$

   Then, $\mathcal{T}$ is also a subspace, and its dimension $\dim(\mathcal{T}) = n - m$.

見背面

二、複選題：請作答於「答案卡」，依題號順序作答 (11., 12., ..., 15.)。
每題有五個選項，至少有一選項為正確。每選對一選項得 +2 分，每選錯一選項得 −1 分 (倒扣)，整題不作答該題得 0 分。例：正確答案為 (A)(B)，若答 (A)(C)(D)，該題得分為 $2+(-1)+(-1)+(-1)+2=1$ 分；若答 (C)(E)，該題得分為 $(-1)+(-1)+(-1)+2+(-1)=-2$ 分。

11. (10%) Consider a matrix

$$M = \begin{bmatrix} 1 & -5 & 2 & -1 \\ -1 & 1 & -5 & 2 \\ 2 & -1 & 1 & -5 \\ -5 & 2 & -1 & 1 \end{bmatrix}. \qquad (\dagger)$$

Which of the following is an eigenvalue of this matrix M?

(A) 1　(B) −3　(C) 9　(D) $-1+2\mathrm{j}$　(E) $-1+4\mathrm{j}$

12. (10%) Consider the matrix M in (†). It is known that M has four distinct eigenvalues, and let us denote the four corresponding eigenvectors as $v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}$.

Which of the following matrices also has $\{v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}\}$ as its eigenvectors?

(A) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$　(B) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$　(C) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$　(E) $\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$

13. (10%) Consider the subspace $\mathcal{S} = \mathrm{span}(\mathcal{B})$ spanned by the following elements in the vector space $\mathsf{M}_{2\times 2}(\mathbb{C})$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \right\} \qquad (\ddagger)$$

Which of the following does not belong to $\mathcal{S}$?

(A) $\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$　(B) $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$　(C) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$　(D) $\begin{bmatrix} 1-j & j \\ 1+j & 1-j \end{bmatrix}$　(E) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

14. (10%) Let $\mathcal{S} = \mathrm{span}(\mathcal{B})$ be a subspace of the vector space $\mathsf{M}_{2\times 2}(\mathbb{C})$ and $\mathcal{B}$ is defined as in (‡). Consider the following mappings:

$$T_1 : \mathcal{S} \to \mathsf{M}_{2\times 2}(\mathbb{C}),\ X \mapsto T_1(X) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} X$$

$$T_2 : \mathcal{S} \to \mathsf{M}_{2\times 2}(\mathbb{C}),\ X \mapsto T_2(X) = X \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$T_3 : \mathcal{S} \to \mathsf{M}_{2\times 2}(\mathbb{C}),\ X \mapsto T_3(X) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} X \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Choose the correct statement(s) from the following.

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(A) Both T_1 and T_2 are linear transforms from $\mathcal{S}$ to $\mathsf{M}_{2\times 2}(\mathbb{C})$.
(B) T_3 is a linear transform from $\mathcal{S}$ to $\mathsf{M}_{2\times 2}(\mathbb{C})$.
(C) The range of T_1 is a subspace of $\mathsf{M}_{2\times 2}(\mathbb{C})$ and its dimension is 2.
(D) The range of T_2 is a subspace of $\mathsf{M}_{2\times 2}(\mathbb{C})$ and its dimension is 2.
(E) The range of T_3 is a subspace of $\mathsf{M}_{2\times 2}(\mathbb{C})$ and its dimension is 2.

15. (10%) For $n > 1$, let $\mathcal{S}_n$ denote the set of all symmetric matrices and $\mathcal{T}_n$ denote the set of all skew-symmetric matrices, that is, $\mathcal{T}_n = \{A \mid A \text{ is a } n \times n \text{ matrix such that } A^T = -A\}$. Let U be a $n \times n$ matrix such that $U^T U = I_n$. Furthermore, let

$$\mathcal{W}_U = \{M \mid M \text{ is a } n \times n \text{ matrix such that } U^T M U \text{ is a diagonal matrix}\}.$$

For a $n \times n$ matrix A, let A_U denote the unique matrix B in $\mathcal{W}_U$ such that $\sum_{i=1}^{n} \sum_{j=1}^{n} (A_{i,j} - B_{i,j})^2$ is minimized.

Choose the correct statement(s) from the following.

(A) $\mathcal{S}_n$ is a subspace of $\mathsf{M}_{n\times n}(\mathbb{R})$ and its dimension is n.
(B) $\mathcal{T}_n$ is a subspace of $\mathsf{M}_{n\times n}(\mathbb{R})$ and its dimension is n.
(C) $\mathcal{W}_U$ is a subspace of $\mathsf{M}_{n\times n}(\mathbb{R})$ and its dimension is n.
(D) Let

$$U = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Then, the first diagonal entry of $U^T A_U U$ is $2 + \frac{\sqrt{3}}{2}$.

(E) Let

$$U = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Then, the second diagonal entry of $U^T A_U U$ is 0.