※ 注意：請於試卷內之「非選擇題作答區」依序作答，並應註明作答之大題及小題題號。

※ Please show the detailed calculation process for the questions whenever necessary.

※ If the answers are with decimal numbers, please round to the second decimal place, e.g., 99.37 or 2.43%.

1. (10%) Find k such that $f(x) = kx^2$ is a probability density function over the interval [2, 5]. Then write the probability density function.

2. (10%) Find the area of the region E bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

3. (10%) Solve $\frac{dy}{dx} + \frac{y}{x} = e^{2x}$.

4. (10%) Suppose that the real-valued function $f: I \to R$ is integrable, where $I = [a, b] \times [c, d]$ is a rectangle in the plane R^2. To use Fubini's Theorem to calculate the integral $\int_I f$, what conditions are required? State and explain Fubini's Theorem in detail.

5. (10%) What are metric spaces? When is a mapping between two metric spaces called continuous?

6. (10%) In February 2021, an automobile manufacturer sells cars in America, Europe, and Asia, charging a different price in each of the three markets. The price function for cars sold in America is $p = 20 - 0.2x$ (for $0 \le x \le 100$), the price function for cars sold in Europe is $q = 16 - 0.1y$ (for $0 \le y \le 160$), and the price function for cars sold in Asia is $r = 12 - 0.1z$ (for $0 \le z \le 120$), all in thousands of dollars, where x, y, and z are the numbers of cars sold in America, Europe, and Asia, respectively. The company's production cost function is $C = 22 + 4(x + y + z)$ thousand dollars.
(a) (2%) Find the company's profit function $P(x, y, z)$, which should be the sum of the revenues from different markets minus production costs, and each revenue is price times quantity.
(b) (8%) Suppose the automobile manufacturer's productivity capacity is 100 cars in this month. How many cars should be sold in each market to maximize profit and what is the maximized profit?

7. (10%) A company's marginal cost function is $MC(x) = xe^{-x/2}$ and fixed costs are 200, where x is a nonnegative real number that denotes the amount of units produced. Find its cost as a function of x. (Hint: Marginal cost is the cost added by producing one additional unit of a product or service.)

8. (30%) Consider a power call or put option with the payoff at the maturity T as

$$C_T = \max(S_T^i - X^i, 0) \text{ and } P_T = \max(X^i - S_T^i, 0),$$

respectively, where S_t is the stock price at time t, i is a positive-integer exponent, and X^i denotes the strike price. Under the Black-Scholes framework, their respective value functions today ($t = 0$) are

$$C_0 = \Pi S_0^i N(d_i) - \exp(-rT) X^i N(d_0),$$
$$P_0 = \exp(-rT) X^i N(-d_0) - \Pi S_0^i N(-d_i),$$

where $\Pi = \Pi(i, r, \sigma, T) = \exp(-rT + irT + (i^2 - i)\sigma^2 T/2)$, $d_i = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)T + i\sigma^2 T}{\sigma\sqrt{T}}$, $d_0 =$

$\frac{\ln\left(\frac{S_0}{X}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$, r is the risk-free interest rate, σ is the stock price volatility, and $N(\cdot)$ is the cumulative distribution function of the standard normal distribution defined as

$$N(d)=\int_{-\infty}^{d} n(x)dx=\int_{-\infty}^{d}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx,$$

where $n(\cdot)$ is the probability density function of the standard normal distribution.

(a) (10%) Prove that $\Pi S_0^i n(d_i)=\exp(-rT)X^i n(d_0)$.

(b) (8%) Derive and express $\frac{\partial C_0}{\partial S_0}$ and $\frac{\partial P_0}{\partial S_0}$ as $AN(B)$ and $CN(D)$, respectively. What are A, B, C, and D?

(c) (8%) Derive and express $\frac{\partial^2 C_0}{\partial {S_0}^2}$ and $\frac{\partial^2 P_0}{\partial {S_0}^2}$ as $EN(F)+Gn(H)$ and $IN(J)+Kn(L)$, respectively. What are E, F, G, H, I, J, K, and L?

(d) (4%) In what condition does $\frac{\partial^2 C_0}{\partial {S_0}^2}$ equal $\frac{\partial^2 P_0}{\partial {S_0}^2}$?

試題隨卷繳回