1. **(25 points)** Enjoy the beauty of the simplex method. Let us solve the following linear programming problem by the simplex method **without** tabular forms. **(Please DO NOT use tabular forms to solve it.)**

$$\text{Maximize } z = 4x_1 + 5x_2$$
subject to
$$6x_1 + 4x_2 \le 24$$
$$x_1 + 2x_2 \le 6$$
$$-x_1 + x_2 \le 1$$
$$x_2 \le 2$$
$$x_1, x_2 \ge 0$$

(a) (5 points) Let us introduce the slack variables x_3, x_4, x_5, x_6 to constraints 1, 2, 3, and 4, respectively so that the objective function and equations, which are called ***dictionary***, can be written as

$$x_3 = 24 - 6x_1 - 4x_2$$
$$x_4 = 6 - x_1 - 2x_2$$
$$x_5 = 1 + x_1 - x_2$$
$$x_6 = 2 - x_2$$
$$z = 0 + 4x_1 + 5x_2.$$

We let x_3, x_4, x_5, x_6 in the above dictionary be basic variables, and x_1 and x_2 be the nonbasic variables. What is the initial feasible solution $(x_1, x_2, x_3, x_4, x_5, x_6, z)$ based on the above dictionary in (a) if both nonbasic variables are set as zero?

(b) (5 points) There are totally six variables, $x_1, x_2, x_3, x_4, x_5, x_6$, with only four equations. Four out of six variables can be expressed by the remaining two variables. For example, x_3 is expressed by x_1 and x_2. Similarly, x_4, x_5, x_6 are expressed by x_1 and x_2 as well. Since we would like to increase z, let us increase x_1 from zero. To keep the feasibility, what is the maximum value we are able to increase the value in x_1?

(c) (5 points) In (a), the current basis is composed of x_3, x_4, x_5, x_6. Suppose x_1 enters the basis and x_3 leaves the basis. What is the dictionary at the next iteration? In other words, please complete the followings.

$$x_1 = ? + ?\,x_2 + ?\,x_3$$
$$x_4 = ? + ?\,x_2 + ?\,x_3$$
$$x_5 = ? + ?\,x_2 + ?\,x_3$$
$$x_6 = ? + ?\,x_2 + ?\,x_3$$
$$z = ? + ?\,x_2 + ?\,x_3$$

(d) (5 points) Following (c), if optimal, justify it. If not, what is the dictionary at the next iteration?

(e) (5 points) What is the optimal dictionary of this problem? In other words, please complete the followings in the final dictionary.

$$x_? = ? + ?\,x_? + ?\,x_?$$
$$x_? = ? + ?\,x_? + ?\,x_?$$
$$x_? = ? + ?\,x_? + ?\,x_?$$
$$x_? = ? + ?\,x_? + ?\,x_?$$
$$z = ? + ?\,x_? + ?\,x_?$$

2. **(10 points)** Reconsider the linear programming problem in Question 1.

$$\begin{aligned} &\text{Maximize } z = 4x_1 + 5x_2 \\ &\text{subject to} \\ &\quad 6x_1 + 4x_2 \le 24 \\ &\quad x_1 + 2x_2 \le 6 \\ &\quad -x_1 + x_2 \le 1 \\ &\quad x_2 \le \cancel{2}\ \ 3 \\ &\quad x_1, x_2 \ge 0 \end{aligned}$$

It turns out that the right-hand side of the fourth constraint is 3 instead of 2. In addition, somehow the maximum number of equations the computer is able to handle is three so let us consider this problem as follows:

$$\begin{aligned} &\text{Maximize } z = 4x_1 + 5x_2 \\ &\text{subject to} \\ &\quad 6x_1 + 4x_2 \le 24 \\ &\quad x_1 + 2x_2 \le 6 \\ &\quad -x_1 + x_2 \le 1 \\ &\quad x_1 \ge 0, 0 \le x_2 \le 3, \end{aligned}$$

where $0 \le x_2 \le 3$ can be viewed as the explicit bound on the individual variable.

(a) (5 points) Only x_3, x_4, x_5 are introduced as the slack variables for the first, second, and third constraints. What is the criteria of the value of x_2 if x_2 is a nonbasic variable?

(b) (5 points) What is the optimal dictionary?

3. **(15 points)** Consider the following linear programming problem.

$$\begin{aligned} \text{Maximize} \quad & 4x_1 + 5x_2 + x_3 + 3x_4 - 5x_5 + 8x_6 \\ \text{subject to} \quad & x_1 - 4x_3 + 3x_4 + x_5 + x_6 \le 1 \\ & 5x_1 + 3x_2 + x_3 - 5x_5 + 3x_6 \le 4 \\ & 4x_1 + 5x_2 - 3x_3 + 3x_4 - 4x_5 + x_6 \le 4 \\ & -x_2 + 2x_4 + x_5 - 5x_6 \le 5 \\ & -2x_1 + x_2 + x_3 + x_4 + 2x_5 + 2x_6 \le 7 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \ge 0. \end{aligned}$$

(a) (5 points) Let the dual variables corresponding to the first to fifth constraints be y_1, y_2, y_3, y_4, y_5. What is the dual problem?

(b) (10 points) Someone tells us that the optimal solution is $x_1 = 0, x_2 = 0, x_3 = \frac{5}{2}, x_4 = \frac{7}{2}, x_5 = 0, x_6 = \frac{1}{2}$. Use the complementary slackness to obtain the value of dual variables.

4. **(10 points)** When one person is infected with c-virus and interacts with another one who is healthy, there is a probability $p = 0.2$ to get the healthy one infected. Two people in the community are selected randomly at any single period of time and interact with each other. Let x_n be the number of infective people at time n and there are five people in the community. Given $x_0 = 1$, what is the mean time to infect everyone in the community?

5. **(20 points)** Clients come to the local post office following a Poisson process with rate λ. Each client has a complex business to do in the post office. Assuming the complexity of the business of an individual client, denoted as c, is following a uniform distribution over $(0, 1)$. The service time of the client with complexity c in the post office is a random variable with mean $5 + 3c$ and variance 4. Calculate

(a) (10 points) the average time a client spends in the post office?

(b) (10 points) the average time a client having complexity c spends in the system?

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6. **(20 points)** Marie is running a small mobile shop and now selling the very popular ePhone. In such a small town, the daily demand for ePhone is limited and following a probability distribution shown in the table below. Since the shop is with a little capital, Marie can keep at most 2 ePhones in inventory every day. When the inventory level is low at the end of the day, she makes an order to the phone suppliers. Assuming the replenishment is instant, i.e., when Marie makes the order at the end of today, the ordered phone(s) will arrive at the beginning of tomorrow. There are two types of costs associated with each order: the fixed cost of 400 NTD for the truck; the variable cost of 100 NTD/phone. The holding cost for an unsold phone in the inventory is 100 NTD. If the demand exceeds the available phones in inventory during the day, the backordered cost is incurred with the penalty of 1000 NTD per phone.

Daily Demand Quantity	**0**	**1**	**2**
Probability	0.2	0.5	0.3

(a) (10 points) Marie is now using the ordering policy: if there are no phones in inventory at the end of the day (including backorder), 2 phones are ordered; otherwise, Marie makes no order. Let the states indicate the number of phones in inventory at the end of the day. Help Marie calculate the long-run expected cost per day for this ordering policy.

(b) (10 points) Find and demonstrate a better ordering policy than the one Marie uses, i.e., the policy with a lower cost.

(For your convenience: $\begin{bmatrix} 1.78 & 0.75 & 0.88 & 1.25 \\ 0.75 & 1.78 & 0.25 & 0.88 \\ 0.88 & 0.25 & 1.98 & 0.75 \\ 1.25 & 0.88 & 0.75 & 1.98 \end{bmatrix}^{-1} = \begin{bmatrix} 1.1625 & -0.18 & -0.2875 & -0.545 \\ -0.18 & 0.75 & 0.08 & -0.25 \\ -0.2875 & 0.08 & 0.6625 & -0.105 \\ -0.545 & -0.25 & -0.105 & 1 \end{bmatrix}$)

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