1. (15 points) Let S be a set of n elements. Let A, B be two different subsets of S chosen uniformly at random. What is the probability that A is a subset of B? Show your derivation.

2. (10 points) Solve the following recurrence (show your derivation):

$$\begin{aligned} a_0 &= 2, \\ a_1 &= 1, \\ a_n &= 5a_{n-1} - 6a_{n-2} + 2^n, \text{ for all } n \geq 2. \end{aligned}$$

3. (15 points) Find a positive integer p such that $\frac{(p+13)!}{p!13!} \equiv 7 \pmod{13}$ or show that such an integer does not exist. Prove the correctness of your answer.

4. (35 points) For each of the following statements, determine whether it is true of false. No explanation is needed. You get +5 points for every correct answer and -6 points for every incorrect one. (0 points if you do not answer.)

   (a) $\exists x(P(x) \wedge Q(x)) \equiv \exists x P(x) \wedge \exists x Q(x)$.

   (b) In propositional logic, $\{\oplus, \leftrightarrow\}$ is a functionally complete set.

   (c) If A and B are two countably infinite sets, then $|A| = |B|$.

   (d) If S is an infinite set, then 2^S must be uncountable.

   (e) If a relation R is transitive, then R^2 must also be transitive.

   (f) The set $\{(f_1(n), f_2(n)) \mid f_1(n) \in O(f_2(n))\}$ is a partial ordering on the set of all positive functions $f : \mathbb{N} \to \mathbb{R}^+$.

   (g) If R_1 and R_2 are two different relations defined on set A, then the (directed) graphs representing R_1 and R_2 must not be isomorhpic.

5. (10 points) Let $G = (V, E)$ be a simple planar undirected graph with every vertex having degree 5. Is it true that G must have at least 12 vertices? Prove your answer.

6. (15 points) If a graph G has chromatic number k, but every graph G' resulting from removing one edge from G has chromatic number at most $k-1$. Is it always true that every vertex in G has degree at least $k-1$? Prove your answer. Recall that the chromatic number of a graph is the minimum number of colors required to color all vertices such that adjacent vertices have different colors.