(1) (20 points) Let V_1 be the $\mathbb{R}$-*linear span* of functions: $\sin^i x \cdot \cos^j x$, $i, j = 0, ..., n$. Let V_2 be the $\mathbb{R}$-*linear span* of functions: $\sin kx.\ \cos kx$, $k = 0, ..., n$. Determine the dimensions of V_1 and V_2 and prove your assertion. Is it true that $V_1 = V_2$? Prove or disprove it.

(2) (15 points) Let $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a *linear transformation* and let id be the identity map sending every $v \in \mathbb{R}^n$ to v. Prove that there exist $C > 0$ such that for all $t \in \mathbb{R}$, $|t| > C$, the map $id + t \cdot \varphi$ is surjective.

(2) (15 points) Let $A := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $B := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$,

$$V = \{v \in \mathbb{C}^4 \mid A \cdot v = \lambda_a \cdot v, B \cdot v = \lambda_b \cdot v, \text{ for some } \lambda_a, \lambda_v \in \mathbb{C}\}.$$

Find a basis of V.

(4) (15 points) Let A be an $n \times n$ *diagonal* matrix with diagonal entries $A_{11}, ..., A_{nn}$. Show that the *linear span* W of A^k, $k = 0, 1, ...$, is of dimension n if and only if $A_{ii} \neq A_{jj}$ for different i and j.

(5) (15 points) Suppose φ and g are $\mathbb{R}$-linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $g \circ \varphi = \varphi^2 \circ g$ and g is *injective*. Show that φ and φ^2 have the same kernel (null-space), image, *eigenvalues* and *eigenspaces*.

(6) Prove or disprove the following statements (10 points for each).
Let $Q : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a quadratic form.

(a) Let $\mathbb{Z}^n \subset \mathbb{R}^n$ denote the subset consisting of vectors with integer coordinates. Then Q is *positive definite* if and only if $Q(v) > 0$ for all $v \in \mathbb{Z}^n$.

(b) There is some $n \times n$ matrix A such that $Q(v) = v^t \cdot A^t \cdot A \cdot v$, for all $v \in \mathbb{R}^n$. Here, B^t denotes the *transpose* of B.

試題隨卷繳回