國立臺灣大學 108 學年度碩士班招生考試試題

(1) [15 points] Let $\gamma(s) : I \to \mathbb{R}^3$ be a curve parametrized by arc-length. Suppose that $\gamma''(s) \neq 0$. Denote by $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ its Frenet frame. The *binormal line* at $\gamma(s)$ is the line passing through $\gamma(s)$ with direction $\mathbf{B}(s)$.

Suppose that $\gamma(I)$ lies in a sphere, and that all its binormal lines are tangent to this sphere. Show that γ is an arc of the great circle of that sphere.

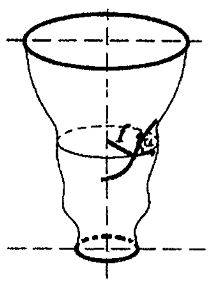
(2) [20 points] Suppose that S is a surface of revolution:

$$S = \{(f(s)\cos\theta, f(s)\sin\theta, g(s)) \in \mathbb{R}^3\}$$

where f is a positive, smooth function. For any curve

$$\gamma(t) = (f(s(t))\cos\theta(t), f(s(t))\sin\theta(t), g(s(t)))$$

on S, let $\alpha(t)$ be the angle between $\gamma'(t)$ and the corresponding latitudinal circle; see the picture. (Recall that the latitudinal circle on a surface of revolution is the circle given by $s =$ constant.)



Suppose that $\gamma(t)$ is a *geodesic* of S. Prove that $f(s(t))\cos\alpha(t)$ is a constant.

(3) [20 points] Let S be the regular surface in $\mathbb{R}^3$ given by

$$S = \left\{ \left(x, y, \log\frac{\cos x}{\cos y} \right) \in \mathbb{R}^3 \,\middle|\, -\frac{\pi}{2} < x, y < \frac{\pi}{2} \right\}.$$

Calculate the Gaussian curvature and mean curvature of S.

(4) [20 points] For a regular surface $\mathbf{x} = \mathbf{x}(u,v)$, denote by $E(u,v)$, $F(u,v)$ and $G(u,v)$ the coefficients of its first fundamental form, and by $e(u,v)$, $f(u,v)$ and $g(u,v)$ the coefficients of its second fundamental form.

Does there exist a regular surface with $E = 3$, $F = 1$, $G = 2$ and $e = v$, $f = u$, $g = \cos u$? Justify your answer.

(5) Let S be closed (compact without boundary) regular surface in $\mathbb{R}^3$ of genus 1. Denote by $K(p)$ the Gaussian curvature of S at p. Prove that

(a) [15 points] there exists $q_0 \in S$ such that $K(q_0) < 0$;

(b) [10 points] there exists $q_1 \in S$ such that $K(q_1) = 0$.