1. (10 points)
Let $I = [a,b]$. Construct an explicit C^∞ function $f(x)$ on $\mathbb{R}$ such that $f(x) > 0$ for any $x \in (a,b)$ and $f(x) = 0$ otherwise. (you have to prove your function is smooth)

2. (20 points)
Suppose $f(x)$ is a continuous function on $[a,b]$ and $\phi(x)$ is increasing and of C^1 on $[a,b]$. Show that there exists a point $c \in [a,b]$ such that

$$\int_a^b f(x)\phi(x)dx = \phi(a)\int_a^c f(x)dx + \phi(b)\int_c^b f(x)dx.$$

3. (20 points)
Let $f(x,y) = \frac{x^2y}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$.
(a) Show that f is continuous.
(b) Show that the directional derivatives $\partial_u f(0,0)$ all exist and compute them.
(c) Is f differentiable at $(0,0)$? Justify your answer.

4. (20 points)
Define $g(x,y,z) = (x^2+y^2+z^2)^{-\frac{1}{2}}$ for $(x,y,z) \neq (0,0,0)$.
(a) Compute the value of $\iint_S (\partial g/\partial n)dA$ where S is the unit sphere centered at origin and n is its unit outward normal vector.
(b) Let R be a region with piecewise smooth boundary whose interior contains the origin. Show that the value $\iint_{\partial R}(\partial g/\partial n)dA$ is a fixed constant, i.e. independent of R.

5. (20 points)
(a) Suppose $f(x)$ is Riemann integrable on $[a,b]$. Does $\lim_{n\to\infty}\int_a^b f(x)\sin(nx)dx$ exist ? Prove or disprove your result.
(b) Let $\{c_n\}$ be the sequence $\{n^2\}$. Does $\lim_{n\to\infty}\int_1^2 \cos^2(nx+c_n)dx$ exist ? Prove or disprove your result. If the limit exists, find the value.

6. (10 points)
Define the sequence $\{e_n\}$ by $e_1 = 1$ and $e_{n+1} = (n+1)(e_n+1)$.
(a) Let $s_n = \sum_{k=0}^{n}\frac{1}{k!}$. Show that $e_n = n!s_{n-1}$.
(b) Show that $s_n = \prod_{k=1}^{n}\frac{e_k+1}{e_k}$ and find the value of $\prod_{k=1}^{\infty}\frac{e_k+1}{e_k}$.

試題隨卷繳回