1. A subgroup H of S_n is said to be *transitive* if for all $i, j \in \{1, \ldots, n\}$, there exists $\sigma \in H$ such that $\sigma(i) = j$.
   (a) (10%) Prove that if H is a transitive subgroup of S_n, then $|H|$ is divisible by n.
   (b) (10%) Find all transitive subgroups of S_4.

2. (a) (10%) Prove that every group of order 15 is cyclic.
   (b) (10%) Show that there are exactly 4 nonisomorphic groups of order 30.

3. (a) (10%) Let R be a commutative ring. Prove that the set

$$N(R) = \{a \in R : a^n = 0 \text{ for some } n > 0\}$$

is an ideal of R.
   (b) (5%) Determine $N(\mathbb{Z}/360\mathbb{Z})$.

4. (15%) Prove that if p is a prime congruent to 5 or 7 modulo 8, then $\mathbb{Z}[\sqrt{-2}]/\langle p \rangle$ is a field, where $\langle p \rangle$ denotes the ideal of $\mathbb{Z}[\sqrt{-2}]$ generated by p.

5. (15%) Let F be a finite field. Prove that the product of all nonzero elements in F is equal to -1.

6. (15%) Let $f(x)$ be an irreducible polynomial of degree 3 over $\mathbb{Q}$ and E be its splitting field over $\mathbb{Q}$. Let $\alpha_1, \alpha_2, \alpha_3$ be the three zeros of $f(x)$ and set

$$\Delta = (\alpha_1 - \alpha_2)^2(\alpha_2 - \alpha_3)^2(\alpha_3 - \alpha_1)^2.$$

Prove that $\mathrm{Gal}(E/\mathbb{Q})$ is isomorphic to the cyclic group of order 3 if Δ is a square in $\mathbb{Q}$ (i.e., $\Delta = r^2$ for some $r \in \mathbb{Q}$) and is isomorphic to S_3 if Δ is not a square in $\mathbb{Q}$.

試題隨卷繳回