※ 注意：請於試卷內之「非選擇題作答區」作答，並應註明作答之題號。

一、(20%) **是非題**（共 10 小題，每題 2 分。答錯每題倒扣 1 分至本大題 0 分為止；不作答者，得 0 分。）

1. Suppose there are n people born between the year 0 A.D. and the year 2003 A.D. It is possible to sort all of them by birthdate in $o(n \lg n)$ time.
2. The following array is a min-heap.

   2 5 9 8 10 13 12 22 50
3. If $f(n) \in \Omega(g(n))$ and $g(n) \in O(f(n))$, then $f(n) \in \Theta(g(n))$.
4. In a dynamic programming solution, its space complexity is always at least as large as the number of unique subproblems.
5. Given a directed acyclic graph $G = (V, E)$ where the edge weights are of real numbers, Dijkstra's algorithm can be applied to solve the single-source shortest-path problem in G.
6. Given an unweighted, undirected graph $G = (V, E)$ and two vertices $u, v \in V$, breadth-first search can be applied to find the shortest path between u and v or whether no such path exists.
7. If all edges in the graph have integer capacities, then there exists a maximum flow in which all flows are integers.
8. For a k-bit binary counter that supports two operations: INCREMENT and DECREMENT, the worst-case amortized cost of randomly performing n operations is $\Theta(nk)$. (Assume that each bit-flip costs 1 unit of time, and that the value of the counter is always positive and never exceeds $2^k - 1$.)
9. If any problem in *NP* can be solved in linear time, then all problems in *NPC* can be solved in polynomial time.
10. If any problem in *NPC* can be solved in linear time, then *P* = *NP* = *NPC*.

二、(55%) **單選題**（共 22 小題，每題 2.5 分。答錯不倒扣。）

For problems 11-14, choose the appropriate tight bound for the recurrence of $T(n)$. Also note that in this exam, the degree of a node x in a tree indicates the number of children of x.

11. $T(n) = 25 \cdot T(n/5) + n^4 \lg n$

    A. $\Theta(n)$　B. $\Theta(n^2)$　C. $\Theta(n^2 \lg n)$　D. $\Theta(n^4 \lg n)$　E. None of the above
12. $T(n) = T(n/3) + T(2n/3) + n$

    A. $\Theta(\lg n)$　B. $\Theta(n)$　C. $\Theta(n \lg n)$　D. $\Theta(n^2)$　E. None of the above
13. $T(n) = 2 \cdot T(n/2) + (n/\lg n)$

    A. $\Theta(\lg n)$　B. $\Theta(n)$　C. $\Theta(n \lg \lg n)$　D. $\Theta(n \lg n)$　E. None of the above
14. $T(n) = 4 \cdot T(n/3) + n^{\log_3 4}$

    A. $\Theta(\lg n)$　B. $\Theta(n)$　C. $\Theta(n^{\log_3 4})$　D. $\Theta(n^{\log_3 4} \lg n)$　E. None of the above
15. The average case time complexity of quick sort over n items is (choose the tightest bound)

    A. $O(n)$　B. $O(n \log^* n)$　C. $O(n \log n)$　D. $O(n^2)$　E. $O(n^2 \log n)$
16. The worst-case time complexity of one push/pop operation in a stack of n items is (choose the tightest bound)

    A. $O(1)$　B. $O(\log^* n)$　C. $O(\log n)$　D. $O(n)$　E. $O(n \log n)$
17. Consider a hash table H with 11 slots. Suppose we use quadratic probing as the collision resolution strategy, where the i-th probe position for a key k is given by the function

    $$h(k, i) = (k + i^2)(\mathrm{mod}\ 11)$$

    Suppose we insert the following 8 keys to the hash table in the exact sequence: $82, 90, 30, 89, 61, 44, 28, 51$. What is the index of the slot storing the key 51?

    A. 0　B. 5　C. 7　D. 9　E. 10
18. What is the maximum height of an AVL tree with 108 nodes? (A tree with only one node has height 0)

    A. 4　B. 6　C. 8　D. 10　E. 12
19. Consider the AVL tree in Figure 1. After inserting key 60, what's the parent of the node with key 87?

    A. 42　B. 60　C. 66　D. 73　E. 94
20. Consider the AVL tree in Figure 1. After deleting key 12, what's the parent of the node with key 54?

    A. 42　B. 66　C. 73　D. 87　E. 94

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21. The worst-case time complexity of n splay operations in a splay tree of n items is (choose the tightest bound)
A. $O(n)$ B. $O(n\log^* n)$ C. $O(n\log n)$ D. $O(n^2)$ E. $O(n^2\log n)$
22. The worst-case time complexity of one delete operation in a splay tree of n items is (choose the tightest bound)
A. $O(1)$ B. $O(\log^* n)$ C. $O(\log n)$ D. $O(n)$ E. $O(n\log n)$
23. In a red-black tree, suppose there is a path with 3 red nodes, then there are at least how many black nodes in the tree?
A. 0 B. 4 C. 7 D. 8 E. 11
24. The number of items in a binomial tree whose root has degree 8 is
A. 9 B. 34 C. 64 D. 128 E. 256
25. The worst-case time complexity of constructing a binomial heap from an unordered array of n items is (choose the tightest bound)
A. $O(n)$ B. $O(n\log^* n)$ C. $O(n\log n)$ D. $O(n^2)$ E. $O(n^2\log n)$
26. How many nodes have degree 5 in a binomial heap with 2019 items?
A. 32 B. 63 C. 64 D. 128 E. 256
27. What is the maximum possible degree of a root in a Fibonacci heap with 256 nodes?
A. 8 B. 9 C. 10 D. 11 E. 16
28. The worst-case time complexity of n decrease key operations in a Fibonacci heap of n items is (choose the tightest bound)
A. $O(n)$ B. $O(n\log^* n)$ C. $O(n\log n)$ D. $O(n^2)$ E. $O(n^2\log n)$
29. The worst-case time complexity of one decrease key operation in a Fibonacci heap of n items is (choose the tightest bound)
A. $O(1)$ B. $O(\log^* n)$ C. $O(\log n)$ D. $O(n)$ E. $O(n\log n)$
30. Let x be a non-root node in a Fibonacci heap. If x has degree 6, at least how many children of x should have degree 3?
A. 0 B. 1 C. 2 D. 3 E. 4
31. Consider disjoint-set union/find algorithms with weighted union and path compression. Assume the disjoint-set forest consists of n items, what is the worst-case time complexity of n find/union operations? (choose the tightest bound)
A. $O(n)$ B. $O(n\log^* n)$ C. $O(n\log n)$ D. $O(n^2)$ E. $O(n^2\log n)$
32. Following the previous question, the worst-case time complexity of one find operations is (choose the tightest bound)
A. $O(1)$ B. $O(\log^* n)$ C. $O(\log n)$ D. $O(n)$ E. $O(n\log n)$

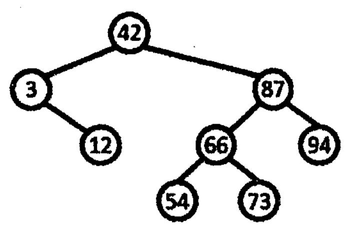

Figure 1: AVL Tree

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三、(5%) **複選題**（共 1 小題，每題 5 分。每題至少有一個正確選項。每答對一個選項得 1 分；每答錯一個選項倒扣 1 分至本大題 0 分為止；不作答者，得 0 分。）

33. Suppose $f(n)$ is positive function of n, which of the following statement(s) are true?

A. If $f(n)$ is $\Omega(n^2)$ and that $\frac{f(n)}{n^2}$ exists, then $\frac{f(n)}{n^2} > 0$.

B. If $f(n)$ is $O(n^2)$ and that $\frac{f(n)}{n^2}$ exists, then $\frac{f(n)}{n^2} < \infty$.

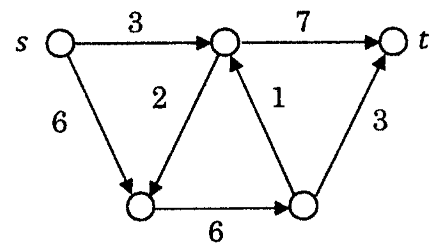
C. If $f(n)$ is $\Theta(n^2)$, then $\frac{f(2n)}{f(n)}$ exists, and $\frac{f(2n)}{f(n)} = 4$.

D. If $f(n)$ is $O(n!)$ then $\log f(n)$ is $O(n \log n)$.

E. If $f(n)$ is $O(\log_2 n)$ then $2^{f(n)}$ is $O(n)$.

四、(20%) **簡答題**（共 2 小題，每題 10 分。所有題目皆應作答於答案卷上。）

1. (a) (3%) What is the maximum $s-t$ flow in the graph below?



(b) (3%) Draw a line that slices the graph into two pieces along the minimum $s-t$ cut.

(c) (4%) Draw the residual graph that results after running the Ford-Fulkerson algorithm to completion on the above graph. Label the edges with their residual capacities.

2. Santa Claus is packing his sleigh with gifts. His sleigh can hold no more than c kilograms. He has n different gifts, and he wants to choose a subset of them to pack in his sleigh. Gift i has utility u_i (the amount of happiness gift i induces in some child) and weight w_i. Define the weight and utility of a *set of gifts* as follows:

- The weight of a set of gifts is the ***sum*** of their weights.
- The utility of a set of gifts is the ***product*** of their utilities.

For example, if Santa chooses two gifts such that $w_1 = 3,\ u_1 = 4$ and $w_2 = 2,\ u_2 = 2$, then the total weight of this set of gifts is 5 kilograms and the total utility of this set of gifts is 8. All numbers mentioned are positive integers and for each gift i, $w_i \le c$. Your job is to devise an algorithm that lets Santa maximize the utility of the set of gifts he packs in his sleigh without exceeding its capacity c.

(a) (3%) A greedy algorithm for this problem takes the gifts in order of increasing weight until the sleigh can hold no more gifts. Give a small example to demonstrate that the greedy algorithm does not generate an optimal choice of gifts.

(b) (7%) Let $H(k,x)$ be the maximal achievable utility if the gifts are drawn from 1 through k (where $k \le n$) and weigh at most x kilograms. Give a recurrence of $H(k,x)$ that can be used in a dynamic program to compute the maximum utility of a set of gifts that Santa can pack in his sleigh. Remember to evaluate the base cases for your recurrence.