Linear Algebra (50%)

1. (20%) Label the following statements as being true or false. (No explanation is needed. Each correct answer gets 2% and each wrong answer gets 0%):
   (a) Let A be an $n \times n$ matrix with real entries. If A is symmetric, then all the eigenvalues of A are real.
   (b) Let A be an $n \times n$ matrix. If A is diagonalizable, then A has n distinct eigenvalues.
   (c) Let A be an $n \times n$ matrix. Then A is singular if and only if 0 is an eigenvalue of A.
   (d) Let $\mathcal{S}$ be a subspace of $\mathcal{R}^n$ and $\mathcal{S}^\perp$ be the orthogonal complement of $\mathcal{S}$. Then $\dim(\mathcal{S}) + \dim(\mathcal{S}^\perp) = n$.
   (e) Let A and B be 2×2 matrices. Then $\text{rank}(AB) = \text{rank}(BA)$.
   (f) Let A and B be $n \times n$ matrices. If A is nonsingular and B is singular, then $A + 2B$ is nonsingular.
   (g) Let A and B be $n \times n$ matrices. If A is similar to B, then A and B have the same eigenvalues.
   (h) If the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent, then the following vectors
   $$2\mathbf{v}_1 + \mathbf{v}_2 + 2\mathbf{v}_3, \qquad \mathbf{v}_2 + \mathbf{v}_3, \qquad \mathbf{v}_2 + 2\mathbf{v}_3$$
   are also linearly independent.
   (i) Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be $n \times 1$ vectors. Then $\text{rank}(\mathbf{v}_1\mathbf{v}_1^T + \mathbf{v}_2\mathbf{v}_2^T) = 2$.
   (j) Let A be an $n \times n$ symmetric matrix. Let $\mathbf{v}$ be an $n \times 1$ vector. If $\mathbf{v}^T A\mathbf{v} = 0$, then $A\mathbf{v} = \mathbf{0}$.

2. Let the matrix $A = \begin{bmatrix} 1 & -3 & 4 \\ -3 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$. Then
   (a) (6%) Find all the eigenvalues of A.
   (b) (6%) Find an orthonormal basis for $\mathcal{R}^3$, consisting of the eigenvectors of A.
   (c) (6%) Find $\mathbf{e}_1^T A^{-1}\mathbf{e}_1$, where $\mathbf{e}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$.
   (d) (2%) Find the nullity of A.

3. Let T be a linear operator on $\mathcal{R}^3$ such that
   $$T\left(\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 10 \\ 4 \\ 2 \end{bmatrix}, \quad T\left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix}.$$
   (a) (4%) Find $T\left(\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T\right)$.
   (b) (6%) Find $T\left(T\left(T\left(T\left(\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T\right)\right)\right)\right)$.

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Probability (50%)

4. The mathematical notion of random variable (RV) is very important in probability theory.
   a) What is the mathematical definition of a RV? (3%)
   b) What is the mathematical definition of a probability function? (3%)
   c) Consider the experiment of throwing a fair 2-sided dice, where the sample space S = { ∘ , ∶ } and Prob({ ∘ }) = Prob({ ∶ }) = 0.5.
   Define a random variable X for this experiment and write down the probability mass function PMF of X, i.e., Prob(X=x) for all real $x \in R$. (5%)

5. You are waiting for a taxi. The inter-arrival time interval between two taxi arrivals is random, say T, and inter-arrival time intervals are independent. You know that T has an exponential distribution and that the probability density function of T is parameterized by $\mu > 0$ as follows:
$$f_T(t) = \begin{cases} \mu e^{-\mu t}, & if\ t \geq 0; \\ 0, & if\ t < 0. \end{cases}$$
   a) You know that the previous taxi arrival is **s** time ago. Let T_R be the remaining time you have to wait for the next taxi arrival. Derive the probability that you will need to wait for no more than a time duration **r**, knowing that the previous arrival was **s** time ago. (You need to give the derivation, not just the answer. 10%)
   b) Now (time 0), you decide not to take a taxi but to count the number of taxi arrivals in a period [0, τ]. Let the number of arrivals in [0, τ] be K, which is a RV. Derive Prob(K = 0). (You need to give the derivation, not just the answer. 5%)

6. X and Y are two random variables with a joint probability density function as
$$f_{XY}(x, y) = \begin{cases} cxy, & 0 \leq x \leq y \leq 2; \\ 0, & otherwise. \end{cases}$$
   a) Derive the conditional cumulative distribution function $F_{X|Y}(x|y)$. (5%)
   b) E[E[X|Y]] = ? (5%)
   c) Is the correlation coefficient $\rho_{XY} = 0$? Explain why. (5%)

7. You are observing a radar signal sequence
$$Y_k = \theta + \omega_k,\ k=1,2,..$$
   where θ is an unknown constant and ω_k is $N(0,1)$ and independent and identical over time index k. Given m observations of Y_k, i.e., given $\{y_1, y_2, \ldots, y_m\}$,
   a) How do you estimate the value of θ ? (5%)
   b) Is your estimate unbiased, i.e., does the mean of your estimate equal θ? Why? (4%)

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