

1. Suppose  $T$  is a function of transformation.

(a) What is the definition of linear transformation? (5%)

(b) Suppose  $T$  is defined as: (5%)

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_3 \\ x_1^3 + 3x_2^2 \end{bmatrix}$$

Is  $T$  linear? If it is linear, find its matrix. If it is not linear, give an example that shows why not.

2. Let  $A$  denote a rotation matrix:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

(a) Find the angle of  $\theta$ . (2%)

(b) Find the characteristic polynomial of  $A$ , and use it to find all eigenvalues of  $A$  or to show that none exist. (4%)

(c) Compute  $A^{2011}$ . (4%)

3. Let  $A$  be the following matrix:

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 2 & 5 \end{bmatrix}$$

(a) Find the eigenvalues of  $A$ . (3%)

(b) Find bases for the eigenspaces of  $A$ . (7%)

(c) Write down an invertible matrix  $P$  and a diagonal matrix  $D$  such that  $A = PDP^{-1}$ . (10%)

4. Let  $A = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ . Find the exponential matrix  $e^A$ . (10%)

5. Consider an  $n \times n$  matrix  $A$  with real entries, which satisfies  $A^2 = -I$ , where  $I$  is the identity matrix.

(a) What are the eigenvalues of  $A$ ? (10%)

(b) Give an example of such  $4 \times 4$  matrix  $A$ . (5%)

(c) Does there exist such a matrix for  $n = 3$ ? (5%)

(d) Can be  $A$  symmetric? Why? (10%)

6. Consider a  $4 \times 4$  square matrix  $A$

$$A = \begin{bmatrix} 387 & 456 & 589 & 238 \\ 488 & 455 & 677 & 382 \\ 440 & 982 & 654 & 651 \\ 892 & 564 & 786 & 442 \end{bmatrix}$$

Calculate the sum of the eigenvalues of  $A$ . (6%)

7. Let  $D$  denote the differential operator; that is,  $D(f(t)) = df/dt$ . Each of the following sets is a basis of a vector space  $V$  of functions. Find the matrix representing  $D$  in each of the following basis.

(a)  $\{1, t, \sin 3t, \cos 3t\}$  (7%)

(b)  $\{e^{5t}, te^{5t}, t^2e^{5t}\}$ . (7%)