1. Let A, B, C are independent events with $(A) = 0.5$, $P(B) = 0.1$, $P(C) = 0.6$. Find the following probabilities:
   (a) At least one of the 3 events occur. (5 points)
   (b) Exactly one of three events occur. (5 points)

2. Suppose that $X_1, X_2, \ldots, X_n$ is a random sample of size n from an Exponential distribution with pdf

$$f(x) = \frac{1}{\theta} e^{-x/\theta}, x \geq 0.$$

   (a) Find the pdf of the minimal order statistic, $K = \min(X_1, X_2, \ldots, X_n)$. (5 points)
   (b) Find an unbiased estimator of θ based on K and show whether the estimator is consistent. (5 points)
   (c) Find a better estimator than the one in part (b). Prove that it is better. (5 points)

3. Suppose that X and Y are independent random variables, each uniformly distributed on the interval (0, 1). Let V=X-Y and W=X+Y. Find the distributions of V and W. (10 points)

4. The independent random variables $X_1, X_2, \ldots, X_n$ have the common distribution

$$P(X_i \leq x|\alpha, \beta) = \begin{cases} 0 & if\ x < 0 \\ \left(\frac{x}{\beta}\right)^{\alpha} & if\ 0 \leq x \leq \beta \\ 1 & if\ x > \beta, \end{cases}$$

   where the parameters α and β are positive.
   (a) Find the sufficient statistic for (α, β). (5 points)
   (b) Find the MLEs of α and β. (5 points)
   (c) Assume that α is known and equal to its MLE, α_0. Find an upper confidence limit for β with 95% confidence level. (5 points)

5. (*4 points*) State "Neyman-Pearson Lemma"
6. (*4 points, 1 points for each*) True or False
   (a) For a test statistic, when the critical value is adjusted to increase the probability of type I error, the probability of type II error always decreases or remains the same.

(b) The p-value is dependent with the level of significance α.

(c) If a test rejects at significance level 0.06, then p-value is less than or equal to 0.06.

(d) The generalized likelihood ratio is never larger than 1.

7. (*18 points, 3 points for each*) If $X_1, X_2, \ldots, X_{16}$ are identical and independent r.v.'s from normal distribution $N(\mu, \sigma^2)$ with pdf as $\frac{1}{\sqrt{2\pi}\sigma} e^{\frac{(x-\mu)^2}{2\sigma^2}}$,where $\sigma^2 = 4$. (When $Z \sim \mathrm{N}(0,1)$, then $P(Z > 1.64) = 0.05$ and $P(Z > 1.96) = 0.025$)

(a) Using Neyman-Pearson Lemma to construct the most powerful (MP) test for testing $\mathrm{H_0}$: $\mu = 0$ against $\mathrm{H_A}$: $\mu = 1$, at level of significance $\alpha = 0.05$.

(b) find the power of the MP test of part (a)

(c) explain that the uniformly most powerful (UMP) test for testing $\mathrm{H_0}$: $\mu = 0$ against $\mathrm{H_A}$: $\mu > 0$, at level of significance $\alpha = 0.05$, is the same as the MP test of the part (a).

(d) explain that the uniformly most powerful (UMP) test for testing $\mathrm{H_0}$: $\mu \leq 0$ against $\mathrm{H_A}$: $\mu > 0$, at level of significance $\alpha = 0.05$, is the same as the MP test of the part (a).

(e) state the "unbiased test at level of significant α".

(f) construct the uniformly most powerful unbiased (UMPU) test for testing $\mathrm{H_0}$: $\mu = 0$ against $\mathrm{H_A}$: $\mu \neq 0$, at level of significant $\alpha = 0.05$.

8. (*12 points*) Let $(X_1, X_2, \ldots, X_6)$ be a sample from $Multinomial(n, p_1, p_2, \ldots, p_6)$ with pmf as $\frac{n!}{x_1!x_2!\ldots x_6!} p_1^{x_1} p_2^{x_2} \ldots p_6^{x_6}$.

If Ω is the set consisting of 6 nonnegative numbers that sum to 1. To test the hypothesis $H_0 \colon \Omega_0 \equiv \{p_1 = p_2 = \cdots = p_6 = 1/6\}$ against $H_A \colon \Omega \backslash \Omega_0$

(a) (*4 points*) what are the dimensions of Ω_0 and Ω, respectively ?

(b) (*3 points*) derive the generalized likelihood ratio statistic Λ.

(c) (*2 points*) What is the asymptotic distribution of $-2\log\Lambda$ under $\mathrm{H_0}$ when n is large?

(d) (*3 points*) show that the $-2\log\Lambda$ is asymptotically equivalent to Pearson's chi-square test: $\sum_{i=1}^{6} \frac{(O_i - E_i)^2}{E_i}$, where O_i and E_i denote the observed and expected counts, respectively.

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9. (*12 points*) Let $X_1, X_2, \dots, X_6$ be independent random variables from Poisson(λ_i) with pmf as $e^{-\lambda_i}\frac{\lambda_i^{x_i}}{x_i!}$, $x_i = 0, 1, 2, \dots.$ If Ω is the set consisting of 6 nonnegative numbers. To test the hypothesis $H_0 \colon \Omega_0 \equiv \{\lambda_1 = \lambda_2 = \cdots = \lambda_6\}$ against $H_A \colon \Omega \backslash \Omega_0$

(a) (*4 points*) what are the dimensions of Ω_0 and Ω, respectively ?

(b) (*3 points*) derive the generalized likelihood ratio statistic Λ.

(c) (*2 points*) What is the asymptotic distribution of $-2\log\Lambda$ under H_0 when *n* is large?

(d) (*3 points*) show that the $-2\log\Lambda$ is asymptotically equivalent to Pearson's chi-square test: $\sum_{i=1}^{6}\frac{(O_i - E_i)^2}{E_i}$, where O_i and E_i denote the observed and expected counts, respectively.

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