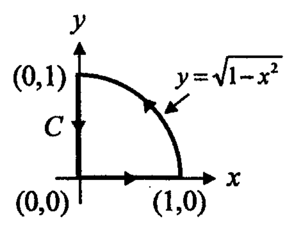
1. (30%). Let $\bar{f}(s) = L[f(x)] = \int_0^\infty e^{-sx} f(x)dx$ be the Laplace transform of $f(x)$.
   (a). (5%). Find out the Laplace transform of the function x^n where n is a positive integer.
   (b). (5%). Let $H(x)$ be the Heaviside unit step function. Show that
   $$L[f(x-a)H(x-a)] = e^{-as}\bar{f}(s),$$
   where $a > 0$ and $\bar{f}(s)$ is the Laplace transform of $f(x)$.
   (c). (10%). Let $y(x)$ satisfy the following differential equation
   $$\frac{d^4y}{dx^4} = \delta(x-1), \qquad 0 < x < 2,$$
   $$y(0)=0, \quad y'(0)=0, \quad y(2)=0, \quad y'(2)=0.$$
   where $\delta(x)$ is the Dirac delta function. Set $y''(0)=\alpha$, $y'''(0)=\beta$. Let $\bar{y}(s)$ be the Laplace transform of the function $y(x)$. Determine $\bar{y}(s)$ in terms of α and β.
   (d). (10%). (Continued with (c)) Find $y(x)$ by determining the inverse Laplace transform of $\bar{y}(s)$. (Note that α and β can be determined by the boundary conditions.)

2. (6%) Let **u** and **v** be two real vectors, where $\mathbf{u} = (u_1, u_2, 1)^T$ and $\mathbf{v} = (0, 2, v_3)^T$. Are there some conditions on u_1, u_2 and v_3, in order to make the following equations valid? If yes, what are the conditions? If no, state your reasons.
   (a) (2%) $\mathbf{u} \bullet \mathbf{v} = \mathbf{0}$
   (b) (2%) $\mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{u} = \mathbf{0}$
   (c) (2%) $\mathbf{u} \times \mathbf{v} - \mathbf{v} \times \mathbf{u} = \mathbf{0}$

3. (24%) Consider a 4×4 matrix A, in the form as indicated, where a, b, c and d are constants. Knowing that the eigenvalues of matrix A are: 1, 1, 4, 5:
   $$\mathbf{A} = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
   (a) (3%) Find the determinant of matrix A.
   (b) (3%) Find the trace of matrix A.
   (c) (3%) Find the determinant of matrix A^2.
   (d) (3%) Find the trace of matrix A^2.
   (e) (3%) Knowing that $a=2, c=1$, find b and d.
   (f) (9%) Continued with (e), find eigenvectors of matrix A.

4. (25%). Suppose that a 30-cm long string has a tension of 100 N and a mass of 75 g. The left end (i.e., $x = 0$) of the string is fixed whereas the right end ($x = 30$ cm) of the string is subject to an external force that yields a transverse displacement of $0.01\sin\omega_d t$ (m) on the right end of the string. The partial differential equation (PDE) for the transverse displacement $u(x,t)$ of the string takes the form $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, where c is the wave speed defined as $\sqrt{\frac{T}{\rho_l}}$, in which T and ρ_l are the tension and linear mass density of the string, respectively.
   (a). (5%). What are the boundary conditions to the PDE?
   (b). (10%). Determine the numerical value of c and solve the transverse displacement $u(x,t)$ along the string in terms of ω_d.
   (c). (5%). Find the resonant frequencies of the string.
   (d). (5%). Assuming the driving frequency of the external force ω_d is 0.8 multiplied by the fundamental resonant frequency (i.e., $\omega_d = 0.8\omega_0$, ω_0 is the fundamental resonant frequency), what is the maximum amplitude of displacement along the string?

5. (15%). Let $w = x^2 y$, and C is the closed curve formed by a quarter circle in the first quadrant.
   (a). (5%). Evaluate $\frac{\partial w}{\partial n}$, i.e., the normal derivative of w, along the circular curve between $(1,0)$ and $(0,1)$.
   (b). (10%). Evaluate $\oint_C \frac{\partial w}{\partial n} ds$.



試題隨卷繳回