

※ 注意：請於試卷內之「非選擇題作答區」依序作答，並應註明作答之大題及小題題號。

1. Consider the following matrix: (20%)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ a & b & c \end{bmatrix}$$

- (a) Give a general condition on the numbers a , b , and c such that the matrix A is singular.
 (b) With your previous condition on a , b , and c , write the matrix A as LU where L and U are some lower and upper triangular matrices.
 (c) With the same previous condition on a , b , and c , find basis vectors for column space, row space, and null space of A .

2. Let A be the following matrix: (20%)

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{bmatrix}$$

Give an explicit and simplified formula for A^n for any positive integer n .

3. (20%)

- (a) Find all solutions to the following system of differential equations:

$$\begin{aligned} y_1' &= -5y_1 + 3y_2 \\ y_2' &= -4y_1 + 2y_2 \end{aligned}$$

- (b) Use your computations from part (a) to find all solutions to the following system of differential equations:

$$\begin{aligned} y_1'' &= -5y_1 + 3y_2 \\ y_2'' &= -4y_1 + 2y_2 \end{aligned}$$

4. Compute the inverse of the matrix (20%)

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 3 \end{bmatrix}$$

- (a) By partitioning the matrix into four 2×2 submatrices.

- (b) By starting with the submatrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and computing the inverse by successively adding rows and columns.

5. Find polynomial solutions of the equation: (20%)

$xy'' + (1-x)y' + ny = 0$ with n a positive integer. Show that these solutions are orthogonal with respect to the weighting function $\exp(-x)$ over the interval $(0, \infty)$.