(Problem 1, 22 points) Consider the following linear programming problem.

$$\text{Maximize } 2x_1 + 3x_2 + 5x_3 + 4x_4$$
$$\text{s.t.} \quad x_1 + 2x_2 + 3x_3 + x_4 \le 5 \qquad (1)$$
$$x_1 + x_2 + 2x_3 + 3x_4 \le 3 \qquad (2)$$
$$x_1, x_2, x_3, x_4 \ge 0.$$

Let introduce $x_5 \ge 0, x_6 \ge 0$ be the slack variables of constraints (1) and (2). That is, we shall define the initial feasible solution as follows:

$$x_5 = 5 - x_1 - 2x_2 - 3x_3 - x_4 \qquad (3)$$
$$x_6 = 3 - x_1 - x_2 - 2x_3 - 3x_4 \qquad (4)$$
$$z = \quad 2x_1 + 3x_2 + 5x_3 + 4x_4, \qquad (5)$$

where x_1, x_2, x_3, x_4 are nonbasic variables as zero value and x_5 and x_6 are basic variables in the initial feasible solution. Please answer the following questions.

(a) What is the current z-value? (2%)

(b) Let us try to increase the value of x_3 from zero to improve the z-row while keeping $x_1 = x_2 = x_4 = 0$. To maintain the feasibility of x_5 and x_6, what is the maximum value of increase in x_3? (2%)

(c) Following (b), what is x_3 in terms of x_1, x_2, x_4, x_6 ? (2%)

(d) Following (c), we have x_3 in terms of x_1, x_2, x_4, x_6. What is x_5 in terms of x_1, x_2, x_4, x_6 ? (2%)

(e) Following (c), what is z in terms of x_1, x_2, x_4, x_6 ? (2%)

(f) What is the z-value in (e)? (2%)

(g) Equations (3) and (4) are essentially two equations with six unknown variables. Please write x_2 in terms of x_1, x_4, x_5, x_6. (2%)

(h) Following (g), please write x_3 in terms of x_1, x_4, x_5, x_6. (2%)

(i) Following (g) and (h), please write z-row in terms of x_1, x_4, x_5, x_6. (2%)

(j) Combining (g) through (i), please write all optimal solutions. (4%)

(Problem 2, 28 points) Consider the following primal problem.

$$\text{Maximize } Z = 2x_1 + 7x_2 + 4x_3 \qquad (6)$$
$$\text{s.t.} \quad x_1 + 2x_2 + x_3 \le 10 \qquad (7)$$
$$3x_1 + 3x_2 + 2x_3 \le 10 \qquad (8)$$
$$x_1, x_2, x_3 \ge 0.$$

(a) Rather than solving the above linear programming problem, we try to get a quick estimate of the optimal value z^* of its objective function. Which of the followings yields a lower bound on the objective function? (2%)

(i) $(x_1, x_2, x_3) = (2,1,1)$

(ii) $(x_1, x_2, x_3) = (1,1,1)$

(iii) $(x_1, x_2, x_3) = (2,2,1)$

(iv) $(x_1, x_2, x_3) = (2,1,2)$

(v) $(x_1, x_2, x_3) = (2,2,0)$

(b) What is the resulting inequality by multiplying constraint (8) by $7/3$? (2%)

(c) What is the upper bound derived by (b) on the objective function? (2%)

(d) Let us construct linear combinations of the constraints. We multiply constraint (7) by y_1, multiply constraint (8) by y_2, and then we add them up. What is the resulting inequality? (2%)

(e) Following (d), suppose that each of two multipliers y_1 and y_2 is nonnegative. What are required constraints so that the resulting inequality in (d) yields an upper bound on the objective function $Z = 2x_1 + 7x_2 + 4x_3$. (4%)

(f) Construct a linear programming model to minimize the upper bound on the objective function of the above problem. (4%)

(g) Use the dual problem to demonstrate that the optimal value of Z for the primal problem is less than or equal to 25. (4%)

(h) Let us assume that x_2 and x_3 should be the basic variables for the optimal solution of the primal problem. What are the value of x_2 and x_3? (2%) Are x_2 and x_3 feasible? (2%) What is the corresponding objective function value? (2%)

(i) Following (h), suppose that $x_4 \cdot y_1 = x_5 \cdot y_2 = x_1 \cdot y_3 = x_2 \cdot y_4 = x_3 \cdot y_5 = 0$, where x_4 and x_5 are slack variables of the primal problem and y_i, $i = 1,\dots,5$ are corresponding dual variables, what are the dual variables $y_1, y_2, \dots, y_5$ associated with the primal problem in (h). (2%)

(Problem 3, 15 points) In Taipei City, each winter is classified as either a cold winter or a rainy winter. Let **U** and **J** be random variables representing the profit of the Umbrella store and the Jacket store, respectively.

U = profit earned by the Umbrella store during a winter

J = profit earned by the Jacket store during a winter

The profits earned by the Umbrella store and the Jacket Store depend on the winter's weather, as shown in the following table.

Type of Winter	Jacket store profit (**J**)	Umbrella store profit (**U**)
Rainy	-1000	1,800
Cold	3,700	-500

(a) Of all winters, 30% are rainy, and 70% are cold. Find cov(**U**,**J**), which is the covariance of **U** and **J**. (5%)

(b) Now, let the type of winter in Taipei follows a Markov chain with the following transition probability matrix.

Transition probability matrix

	Rainy	Cold
Rainy	0.3	0.7
Cold	0.6	0.4

(i) Given the current winter is a cold winter, find the total expected profit of the Jacket store during *the current winter and the next two winters*. (5%)

(ii) Find the *long-run average profit* of the Jacket store. (5%)

(Problem 4, 10 points) NTUIE uses 150 boxes of paper per year for their copy machines and printers. NTUIE orders their paper from a single supplier, and each order costs $100 (shipment cost). The inventory carrying cost (per year) is 20% of the unit price. The supplier delivers orders instantaneously, and the supplier offers the following quantity discount:

Order quantity	Price per box
1-10 boxes	$550
10-25 boxes	$520
25+ boxes	$500

(a) Find the optimal order quantity. (5%)

(b) Find the optimal total annual cost. (5%)

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(Problem 5, 10 points) The NTU Airlines (**NTUA**) sells tickets for the flight between Taipei and Kaohsiung. **NTUA** uses airplanes that hold up to 100 passengers, and the ticket price is $1,500 per passenger. In the past, some passengers purchased the ticket but fail to show up (no-show). The number of no-shows follows a uniform distribution on the interval 10 to 20. (i.e., The number of no-shows is a uniform random variable U(10,20).)

To protect against no-shows, **NTUA** now decides to sell more than 100 tickets. However, passengers with a ticket and unable to board the plane are entitled to compensation of $1,000.

(a) To maximize profit, how many tickets should **NTUA** sell? (10%)

(Problem 6, 15 points) **NTUIE** has $4 million to invest in three projects. The amount of profit earned from project *i* (*i=1,2,3*) depends on the amount of money invested in project *i*, as shown in the following table. The amount invested in a project must be an exact multiple of $1 million.

Amount Invested ($ Millions)	Profit ($ Millions)		
	Project 1	Project 2	Project 3
0	5	4	4
1	8	7	8
2	9	11	9
3	10	13	14
4	12	15	16

(a) Find an investment policy that maximizes the total profit. (10%)

(b) Now, due to insufficient data, the profit from Project 3 is uncertain and equally likely to be a low value or a high value. (The profit from Project 1 and 2 remains unchanged.)
(See the following table for the high/low values.)
(The profit from Project 3 still depends on the amount of money invested.)

Amount Invested ($ Millions)	Profit from Project 3 ($ Millions) **(High/Low values are equally likely.)**	
	High	Low
0	5	3
1	9	7
2	10	8
3	15	13
4	17	15

Find the optimal investment policy and explain your reasoning. (5%)

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