1. Consider the linear differential equation $\ddot{y}+2\dot{y}+2y=f(t)$.
   (a) Solve the initial-value problem and discuss the continuities of $y(t)$ and $\dot{y}(t)$ at $t=0$ under the condition $f(t)=\delta(t)$ and $y(0)=\dot{y}(0)=0$, where $\delta(t)$ is the Dirac delta function. (5%)
   (b) Express the solution of the following initial-value problem in its simplest form.
   $\ddot{y}+2\dot{y}+2y=f(t)$, $y(0)=\dot{y}(0)=0$, where $f(t)$ is any general function. (5%)

2. Find the eigenvalues and their corresponding eigenvectors of the following matrix, and show how to use these eigenvectors to form an orthogonal matrix that can diagonalize the matrix. (15%)

$$\mathbf{A}=\begin{bmatrix}5 & -2 & 0\\ -2 & 6 & -2\\ 0 & -2 & 7\end{bmatrix}$$

3. Which of the following statement is the most accurate? (5%)
   (a) Laplace transform can be considered as a special case or subset of Fourier transform.
   (b) Laplace transform is used for periodic response of a system, while Fourier transform is used for impulse response of a system.
   (c) In principle, Fourier transform is suitable for steady-state signal analysis, while Laplace transform is suitable for transient signal analysis.
   (d) Fourier transform mainly deals with both changing magnitudes and oscillation of a signal.
   (e) If you want to understand what a system does when you flip a light switch, you prefer to use Fourier transform.

4. Expand $f(x)=x(\pi-x)$ at $[0,\pi]$ in a Fourier sine series. Express your answer in the most compact form. Based on your results, find the value of the following summation: (10%)

$$\sum_{n=1}^{\infty}\left[\frac{(-1)^{n-1}}{(2n-1)^3}+\frac{(-1)^{n-1}}{(2n+1)^3}\right]$$

5. Knowing $x(t)$ is a continuous function, solve the following integral equation in which the integration is from 0 to t and the non-homogeneous part is $t\exp(-t)-\sin t$. (10%)

$$\int_0^t ux(u)\cos(t-u)du=te^{-t}-\sin t$$

6. Consider a circular helix $\vec{x}(\theta)=(r\cos\theta,\ -r\sin\theta,\ c\theta)$, where r and c are positive constants.
   (a) Find the arc length for $0\le\theta\le 2\pi$. (3%)
   (b) Find the normal plane at $\theta=0$. (2%)
   (c) Find the unit normal vector $\vec{n}(\theta)$ and unit binormal vector $\vec{b}(\theta)$ as function of θ, i.e. $\vec{n}=\vec{n}(\theta), \vec{b}=\vec{b}(\theta)$. (8%)
   (d) Find the curvature κ as a function of θ, i.e. $k=k(\theta)$. (4%)

7. Consider the helicoid $\vec{x}(u_1,u_2)=(u_1\cos u_2,\ u_1\sin u_2,\ bu_2)$, where b is a constant.
   (a) Find the tangent plane to the helicoid at $u_1=u_2=\pi$. (4%)
   (b) Find the area for $0\le u_1\le a$ and $0\le u_2\le 2\pi$, where a is also a constant. (4%)

8. What is a linear partial differential equation (PDE)? Give an example of non-linear PDE and explain in your view why it is non-linear. (4%)

9. Let $S(x,t)$ denote the displacement of a finite string over $0\le x\le\pi$ with a fixed end at $x=0$ and a free end at $x=\pi$ such that $S(0,t)=0$ and $\partial S/\partial x(\pi,t)=0$. The string starts to vibrate from its initial states $S(x,0)=0$ and $\partial S/\partial t(x,0)=x$ after an external force $F(x)=-x(x-\pi)$. The subsequent string displacement can be described by a 1-D wave equation with a propagation speed of 9. Solve for $S(x,t)$. (7%)

10. (a) Evaluate $\int_C \frac{e^{i\alpha z}}{z^4+1}dz$ where C is a semicircle of radius $R>1$ centered at $z=0$ on the complex plane. (7%)
   (b) Apply 10(a) and Fourier transformation to solve $\frac{d^4 f}{dx^4}+f=g(x)$ for a general driving function $g(x)$, provided that $\frac{d^k f}{dx^k}\to 0$ as $|x|\to\infty$ for integer $k\ge 0$. (7%)

試題隨卷繳回