國立臺灣大學 107 學年度碩士班招生考試試題

1. (15%) You roll a fair die ten times. Let X be the number of *six* in the first four rolls, and let Y be the total number of *six*. Determine $Cov(X, Y)$,

2. (25%) Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be independent random samples from exponential distributions with pdfs

$$f_X(x) = \lambda_1 \exp(-\lambda_1 x), x > 0 \text{ and } f_Y(y) = \lambda_2 \exp(-\lambda_2 y), y > 0$$

respectively. It is known that $\lambda_1 \leq \lambda_2$.

   (a) (15%) Find the MLEs of λ_1, $\hat{\lambda}_1$, and λ_2, $\hat{\lambda}_2$. Give reasons for your answer.

   (b) (10%) Derive the asymptotic distribution of $\hat{\lambda}_1$.

3. (20%) Suppose you fit the model $Exponential(\lambda)$ to data $x_1, \cdots, x_n$ by computing the MLE $\hat{\lambda} = 1/\bar{x}$. Here $\bar{x} = \sum_{i=1}^{n} x_i/n$. But the true distribution of the data $X_1, \cdots, X_n$ are independent and identically distributed with $Gamma(2, 1)$. You may use the fact that if $X \sim Gamma(\alpha, \beta)$, then $E(X) = \alpha/\beta$ and $Var(X) = \alpha/\beta^2$. Show directly, using the Law of Large Numbers, to find the limit of $\hat{\lambda}$ as n goes to the infinity.

4. (20%) A lot consisting of N hard disks contains D defectives, where D is an unknown number. A sample of size n will be selected, without replacement, and the number of defects, X, is obtained.

   (a) (8%) Specify the family of distribution $P_D(X = x)$, $0 \leq D \leq N$.

   (b) (12%) Propose a test for testing $H_0 : D \leq D_0$ versus $H_a : D > D_0$.

5. (20%) Suppose that the random variables $Y_1, Y_2, \cdots, Y_n$ satisfy

$$Y_i = \beta x_i + \epsilon_i$$

for $i = 1, 2, \cdots, n$. Here $x_1, x_2, \cdots, x_n$ are known constants, β is an unknown regression parameter, and $\epsilon_1, \cdots, \epsilon_n$ are independent and identically distributed $N(0, 1)$ random variables.

   (a) (5%) Find the Maximum Likelihood estimator of β, call it $\hat{\beta}$.

   (b) (5%) Construct the Likelihood Ratio Test statistics for testing $H_0 : \beta = 0$ versus $H_a : \beta \neq 0$. and show that the Likelihood Ratio Test statistic can be written in such a way that it involves the data, $Y_1, \cdots, Y_n$, only through $T = \hat{\beta}^2(Y_1, \cdots, Y_n)$.

   (c) (5%) Find the distribution of T under H_0.

   (d) (5%) Suppose that $n = 100$, $\sum_{i=1}^{100} x_i^2 = 10$ and $\sigma^2 = 4$. Give the exact rejection region of size 0.10 Likelihood Ratio Test in terms of T. (Note that $P(Z > 1.645) = 0.05$ when $Z \sim N(0, 1)$.)