1. (20 points) Consider the Enneper's surface parametrized by

$$X(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right)$$

(i) Find the first fundamental form of X. (6 points)
(ii) Find the second fundamental form of X at the point (u,v) = (0,0). (6 points)
(iii) Find the principal curvatures at the point (u,v) = (0,0). (8 points)

2. (30 points) Consider the surface of revolution X given by

$$X(u,v) = (\phi(v)\cos(u), \phi(v)\sin(u), \psi(v))$$

where $\phi(v) > 0$, and $(\phi')^2 + (\psi')^2 = 1$.

(i) Prove that every meridian $X(u_0, v)$ (i.e. u coordinate is constant) is a geodesic, and a parallel $X(u, v_0)$ is a geodesic if and only if $\phi'(v_0) = 0$. (15 points)

(ii) Let $\alpha(s) = X(u(s), v(s))$ be a geodesic on X parametrized by arc length. Let

$e_1 = \frac{X_u}{\|X_u\|}$, $e_2 = \frac{X_v}{\|X_v\|}$,

and let $\theta(s)$ be a smooth function such that

$$\alpha'(s) = \sin\theta(s)e_1(s) + \cos\theta(s)e_2(s)$$

Show that $r(s)\sin\theta(s)$ is constant along α, where r(s) is the distance of $\alpha(s)$ from the z-axis (i.e. the axis of revolution). (15 points)

3. (25 points) Let X(u,v) be a coordinate parametrization of a regular surface X and let $\vec{n} = \frac{X_u \times X_v}{\|X_u \times X_v\|}$ be the unit normal.

Suppose the first fundamental form is of the form

$$\begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix}$$

(i) Prove that

$$X_{uu} + X_{vv} = 2\lambda^2 H\vec{n}$$

where H is the mean curvature. (10 points)

(ii) Prove that the helicoid

$$X(u,v) = (a\sinh(v)\cos(u), a\sinh(v)\sin(u), au)$$

is a minimal surface (i.e. H=0). Here a>0 is a constant. (15 points)

4. (25 points) Suppose that M is an orientable connected compact embedded smooth surface in $\mathbb{R}^3$, containing two hemispheres $S_1 = \{x^2 + y^2 + z^2 = 1, z \le 0\}$ and $S_2 = \{x^2 + (y-5)^2 + (z-5)^2 = 2, y \ge 5\}$ as domains with smooth boundary in M such that the subset $M \setminus S_1 \cup S_2$ is connected.

(i) Using Gauss-Bonnet Theorem, show that the Euler characteristics χ of M and $\overline{M \setminus S_1 \cup S_2}$ are related by

$$\chi(M) = \chi(\overline{M \setminus S_1 \cup S_2}) + 2 \qquad \text{(10 points)}$$

(ii) Suppose that N is another orientable connected compact embedded smooth surface in $\mathbb{R}^3$ such that it intersects the horizontal planes $P_1 = \{z = 0\}$ and $P_2 = \{z = 10\}$ orthogonally at simply closed smooth regular curves α_1 and α_2 on P_1 and P_2 respectively (it means that there is a tangent vector of N at every points on α_i which is orthogonal to P_i.). If the part $N_0 = N \cap \{0 \le z \le 10\}$ is homeomorphic to $\overline{M \setminus S_1 \cup S_2}$ as topological space, express the total Gauss curvature on N_0 (i.e. the integral of Gauss curvature in N_0) in terms of $\chi(M)$. (15 points)

試題隨卷繳回