國立臺灣大學 106 學年度碩士班招生考試試題

(1) (10 points) Let G be a group and G' be its commutator subgroup defined by $G' = \{xyx^{-1}y^{-1} \mid x, y \in G\}$. Let N be a normal subgroup of G. Show that G/N is abelian if and only if $G' \leq N$.

(2) (15 points) Let G be a nonabelian group of order p^3, where p is a prime. Show that the center of G is the commutator subgroup G' of G.

(3) (15 points) Consider the alternating groups A_4 and A_5. Please write down a representative for each conjugacy class in A_4 and A_5 and compute the size of each conjugacy class and the size of the centralizer of each conjugacy class.

(4) (10 points) Prove that if $|G| = 224$ then G is not simple. (You may assume Sylow Theorem.)

(5) (15 points) Let R be a ring with identity. An element $e \in R$ is called an idempotent if $e^2 = e$. Assume e is an idempotent in R and $er = re$ for all $r \in R$. Please prove the following:
(a) Re and $R(1-e)$ are two-sided ideals of R.
(b) $R \simeq Re \times R(1-e)$.
(c) e and $1-e$ are identities for the subrings Re and $R(1-e)$.

(6) (10 points) Let $\zeta = \exp(2\pi i/17)$ be the primitive 17-th root of unity. Show that $\zeta^3 + \zeta^6 + \zeta^7 + \zeta^{10} + \zeta^{11} + \zeta^{12} + \zeta^{14} + \zeta^{15}$ lies in a degree 2 field extension over $\mathbb{Q}$.

(7) (10 points) Let p be a prime and $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree p. Show that if $f(x) = 0$ has exactly 2 roots not lying on the real line, then the Galois group of $f(x)$ is S_p.

(8) (15 points) Let $K = \mathbb{C}(t)$, the field of rational functions in the variable t with complex coefficients. Let $\zeta \in \mathbb{C}$ be a primitive n-th root of unity. Consider the automorphism σ and τ of K over $\mathbb{C}$ defined by $\sigma(t) = t^{-1}$ and $\tau(t) = \zeta t$. Let G be the subgroup in $Aut(K/\mathbb{C})$ generated by σ and τ, and K^G be the fixed field of G.
(a) Show that G is isomorphism to the dihedral group of order $2n$.
(b) Compute the minimal polynomial of t over K^G.
(c) Show that the fixed field K^G is $\mathbb{C}(u)$ for some u in $\mathbb{C}(t)$. Compute u explicitly.

試題隨卷繳回