1. (20%) Label the following statements as being true or false. *(No explanation is needed. Each correct answer gets 2% and each wrong answer gets 0%)*:

   (a) Similar matrices have the same characteristic polynomial.

   (b) If $A^T = -A$, then det $A = 0$.

   (c) Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4]$ and $A' = [\mathbf{a}_1\ \mathbf{a}_4\ \mathbf{a}_2\ \mathbf{a}_3]$. If $A\mathbf{x} = \mathbf{b}$ is consistent, then $A'\mathbf{x} = -2\mathbf{b} + 5\mathbf{a}_2$ is also consistent.

   (d) If λ is an eigenvalue of A^2, then λ is also an eigenvalue of A.

   (e) Let V and W be subspaces of $\mathcal{R}^n$. If $V^\perp = W^\perp$, then $V = W$.

   (f) Let R be the reduced row echelon form of A. Then the reduced row echelon form of $[A\ \ A]$ is $[R\ \ \mathbf{0}]$.

   (g) If A is a nonzero symmetric matrix, then A^2 is also a nonzero symmetric matrix.

   (h) Let A be an $m \times n$ matrix and $\mathbf{b}$ be a vector in $\mathcal{R}^m$. If $A\mathbf{x} = \mathbf{b}$ has a unique solution, then $n \geq m$.

   (i) If two vectors spaces are isomorphic, then they have the same dimension.

   (j) Let $\mathcal{S} = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots, \mathbf{v}_n\}$ be a linearly independent subset of $\mathcal{R}^n$. Let $\mathbf{v}$ and $\mathbf{w}$ be $n \times 1$ vectors. If $\mathbf{v} \cdot \mathbf{v}_i = \mathbf{w} \cdot \mathbf{v}_i$ for $i = 1,\ 2, \ldots,\ n$, then $\mathbf{v} = \mathbf{w}$.

2. Let $W = \text{Span}\ \{\mathbf{v}_1,\ \mathbf{v}_2\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix},\quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

   (a) (5%) Find a basis for $W^\perp$.

   (b) (5%) Find an orthonormal basis for $W^\perp$.

   (c) (5%) Let $\mathbf{v} = [1\ 1\ 1\ 1]^T$. Find a vector $\mathbf{z}$ in $W^\perp$ such that $||\mathbf{v} - \mathbf{z}||$ is minimized.

3. Let T be a linear operator on $\mathcal{P}_2$ defined by

$$T(p(x)) = \big(p(0) + p(1)\big) \ +\ \big(p'(0) + p(1)\big)x \ +\ \big(p(0) + p'(1)\big)x^2,$$

   where $p'(x)$ is the derivative of $p(x)$.

   (a) (6%) Let $\mathcal{B} = \{1,\ x,\ x^2\}$ be a basis for $\mathcal{P}_2$. Find $[T]_\mathcal{B}$, the matrix representation of T with respect to $\mathcal{B}$.

   (b) (9%) Find a basis $\mathcal{B}'$ for $\mathcal{P}_2$ such that $[T]_{\mathcal{B}'}$ is a diagonal matrix.

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4. (20%) Provide answers to the following 4 sub-problems. Mathematical derivation is optional.

   (a) (5%) Consider the trials of testing each IC for being accepted or rejected. Let X be the random variable that represents the minimum number of trials needed until there are $k > 0$ accepted ICs and $F_X(x)$ be its cumultaive distribution function (CDF). Let Y be the random variable that represents the number of accepted ICs in $n > 0$ trials and $F_Y(y)$ be its CDF. Write $F_X(x)$ in terms of $F_Y(y)$.

   (b) (5%) Assume that packet arrivals in a time duration of τ follow the Poisson distribution with expected value λ. It is known that there is one packet arrival in a duration of τ. What is the probability distribution of its arrival time T?

   (c) (5%) Let $X_1, X_2, \ldots, X_n$ be *iid* continuous uniform $(0, 1)$ random variables, where $n > 0$ is a constant. Let

   $$W = \lim_{n\to\infty} n \min(X_1, X_2, \ldots, X_n)$$

   be another random variable. Find the probability density function (PDF) of W.

   (d) (5%) Let $X_1, X_2, \ldots, X_N$ be *iid* Gaussian random variables each with expected value 10 and standard deviation 2, where N is a Poisson random variable with expected value 5. What is the variance of $X_1 + X_2 + \cdots + X_N$?

5. (16%) The theorem of *Continuity of Probability Measure* states that for any increasing or decreasing sequence of events, $\{E_n, n \geq 1\}$,

   $$\lim_{n\to\infty} P[E_n] = P\left[\lim_{n\to\infty} E_n\right],$$

   where $P[\cdot]$ is a probability measure that satisfies the axioms of probability.

   (a) (6%) Prove the theorem of *Continuity of Probability Measure.*

   (b) (10%) What is the consequence of the *Continuity of Probability Measure*? State a theorem whose proof directly relies on the theorem of *Continuity of Probability Measure.* Then, prove the theorem that you stated.

6. (14%) Let X and Y be two jointly distributed random variables with the joint PDF as follows:

   $$f_{X,Y}(x,y) = \begin{cases} 2, & 0 \leq x \leq 1, 0 \leq y \leq x, \\ 0, & \text{otherwise.} \end{cases}$$

   For some reason, samples of random variable X can be observed, but not for random variable Y. It is desired to estimate Y based on the observation of X.

   (a) (7%) Find the best *linear* estimate (i.e. $aX + b$ with a, b being constants to be determined) of Y given X such that the mean square estimation error is minimized. What is the minimum mean square error thus found?

   (b) (7%) Find the best estimate (that is not necessarily linear) of Y given X such that the mean square estimation error is minimized. What is the minimum mean square error thus found?

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