國立臺灣大學 106 學年度碩士班招生考試試題

1. (a). (5%) Given a two-dimensional vector field $\mathbf{F} = F_1\mathbf{i} + F_2\,\mathbf{j} = (\frac{-y}{x^2+y^2})\mathbf{i} + (\frac{x}{x^2+y^2})\mathbf{j}$, show that $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ on the 2-D plane except at the origin where $\mathbf{F}$ is undefined.

   (b). Evaluate the integral $I = \int_\Gamma \mathbf{F} \cdot d\mathbf{r}$ when

   (i). (5%) Γ is the circle $x^2 + y^2 = 2$,

   (ii). (5%) Γ is the square with corners P at (1,−1), Q at (3,−1), R at (3, 1), and S at (1, 1).

   (c). (5%) Comment on the results of (b).

2. Consider the eigenvalue problem $\mathbf{Ax} = \lambda\mathbf{x}$ where $\mathbf{A}[a_{ij}]$ is an $n \times n$ matrix. The characteristic polynomial associated with $\mathbf{A}$ is given as $P_n(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = (-1)^n(\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n)$.

   (a). (5%) Two $n \times n$ matrices $\mathbf{B}$ and $\mathbf{D}$ are called similar if $\mathbf{D} = \mathbf{Q}^{-1}\mathbf{BQ}$, where $\mathbf{Q}$ is a nonsingular $n \times n$ matrix. Show that the eigenvalues of $\mathbf{B}$ and $\mathbf{D}$ are identical.

   (b). (5%) Prove the Cayley-Hamilton theorem for the case $\mathbf{A}$ being a symmetric matrix: $P_n(\mathbf{A}) = 0$ (note that, in fact, this theorem holds for all matrices). Use this theorem to give the formula for computing $\mathbf{A}^{-1}$.

   (c). (5%) Use (b) to compute the inverse $\mathbf{A}^{-1}$ where $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$ and check the result by showing that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

3. (20%). Find the solution $y(x)$ of the following boundary value problem

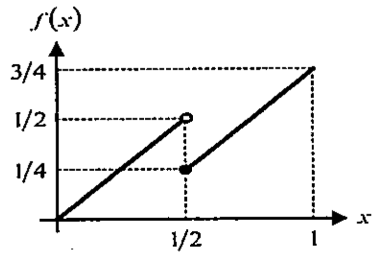
$$\begin{cases} x^2y'' + 2xy' + 4x^2y = 2x \cdot \sin x \cdot \cos x, & 0 < x < \pi/4 \\ y(0): \text{bounded} \quad , & y(\pi/4) = 1 \end{cases}$$

   [hint: you may want to consider a new function of x, say $w(x) = xy(x)$, instead of $y(x)$.]

4. Function $f(x)$ is defined in $0 \le x \le 1$ as shown in the figure.

   (a) (10%) Find the Fourier Sine series representation of $f(x)$.

   (b) (5%) What are the values that the Fourier Since series of $f(x)$ converges to at $x = 0,\ 1/2,\ 1$?



5. (a). (5%) Evaluate $\int_{-\infty}^{\infty} e^{-kx^2}dx$. (Hint: consider $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-k(x^2+y^2)}dxdy$)

   (b). (5%) Define $A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} \cos(\lambda x) e^{-kx^2}dx$. Show that $\frac{dA(\lambda)}{d\lambda} + \frac{\lambda}{2k}A(\lambda) = 0$.

   (c). (5%) Find $A(\lambda)$.

   (d). (5%) Define $\bar{g}(w) = e^{-kw^2t}$. Evaluate the inverse Fourier transform of $\bar{g}(w)$, i.e., find $g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{g}(w)e^{iwx}dw$.

   (e). (10%). Consider a function $u(x,t)$ which is periodic in x with period 1; i.e., $u(0,t) = u(1,t) = \cdots = u(n,t)$ where n is an integer. Suppose $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $t > 0$, $u(x,0) = f(x)$, where $f(x)$ is periodic in x with period 1. Define $\bar{u}(w,t) = \int_{-\infty}^{\infty} u(x,t)e^{-iwx}dx$. Using the technique of Fourier transform applied to the above mentioned partial differential equation, find the general solution of $u(x,t)$ in terms of $f(x)$. In addition, what is the explicit form of $u(x,t)$ if $f(x) = \sin x$?

試題隨卷繳回