GRADUATE ENTRANCE EXAM 2016: LINEAR ALGEBRA

Notation: R is the set of real numbers, and C is the set of complex numbers. If $F = \mathrm{R}$ or C, denote by $M_n(F)$ the $n \times n$ matrices with entries in F.

Problem 1 (10pts). Find all possible $a \in \mathrm{R}$ such that the vectors

$$(1,3,a),\ (a,4,3),\ (0,a,1) \in \mathrm{R}^3$$

are linearly dependent.

Problem 2 (10pts). Find a set of polynomials $p_0(t) = a$, $p_1(t) = b + ct$ and $p_2(t) = d + et + ft^2$ with coefficients $a, b, c, d, e, f \in \mathrm{R}$ so that $\{p_0, p_1, p_2\}$ is an orthonormal set of polynomials with respect to the inner product $\langle f, g\rangle = \int_0^2 f(t)g(t)dt$.

Problem 3 (20pts). Let

$$A = \begin{pmatrix} 1 & -3 & 0 \\ 3 & 4 & -3 \\ 3 & 3 & -2 \end{pmatrix} \in M_3(\mathrm{R}).$$

Find an invertible $P \in M_3(\mathrm{R})$ such that

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & -3 & 1 \end{pmatrix}.$$

Problem 4 (15pts). Let $V = M_3(\mathrm{C})$ be a 9-dimension vector space over C and let

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}.$$

Define the linear transformation $T : V \to V$ by

$$T(B) = ABA^{-1}.$$

Show that T is also diagonalizable.

Problem 5 (20pts). Let $A, B \in M_n(\mathrm{C})$. Suppose that eigenvalues of A and B are all real numbers and that $\operatorname{rank} A = \operatorname{rank} A^2$ and $\operatorname{rank} B = \operatorname{rank} B^2$. If A^3 is similar to B^3 (namely there exists an invertible $P \in M_n(\mathrm{C})$ such that $P^{-1}A^3P = B^3$), prove that A is similar to B.

Problem 6 (25pts). Let A and B be elements in $M_n(\mathrm{C})$. If $A^2B + BA^2 = 2ABA$, show that $(AB - BA)^n = 0$.

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