I. Let $\beta : [0,1] \to \mathbb{R}^3$ be a unite-speed curve with the curvature κ and the torsion τ.

(a) (10%) Show that for the Frenet frame $\{T, N, B\}$

$$N' = -\kappa T + \tau B.$$

(b) (10%) Compute the Frenet frame $\{T, N, B\}$ and κ, τ of the unit-speed helix

$$\beta(s) = (a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}}), \quad a > 0.$$

II. Consider the following smooth Monge patch in $\mathbb{R}^3$:

$$\mathrm{x}(x_1, x_2) = (x_1, x_2,\ f(x_1, x_2)).$$

Show that the image of x is

(a) (10%) flat if and only if

$$f_{x_1x_1} f_{x_2x_2} - (f_{x_1x_2})^2 = 0.$$

(b) (10%) minimal if and only if

$$(1 + f_{x_1}^2) f_{x_2x_2} - 2 f_{x_1} f_{x_2} f_{x_1x_2} + (1 + f_{x_2}^2) f_{x_1x_1} = 0.$$

III. Show that

(a) (10%) The cylinder

$$\mathrm{x}(x_1, x_2) = (r\cos\frac{x_1}{r}, r\sin\frac{x_1}{r}, x_2)$$

is local isometric to the Euclidean space $\mathbb{R}^2$.

(b) (10%) Show that no two of sphere, torus and cylinder are isometric.

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IV. Let $f : \Sigma \to \mathbf{R}^3$ be an isometric immersion of a smooth closed orientable Riemann surface Σ into $\mathbf{R}^3$. We define the Willmore energy

$$W(f) = \int_\Sigma H^2 dA,$$

where $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ is the mean curvature and κ_1, κ_2 are the principal curvatures. Show that

(a) (10%)

$$H^2 - K \geq 0,$$

where $K = \kappa_1 + \kappa_2$ is the Gauss curvature.

(b) (10%)

$$\int_{\Sigma^+} K dA \geq 4\pi,$$

where $K^+ = \max\{K, 0\}$ and $\Sigma^+ = \Sigma|_{K^+}$.

(c) (10%)

$$W(f) \geq 4\pi.$$

(d) (10%) $W(f) = 4\pi$ if and only if Σ is embedded as a round sphere in $\mathbf{R}^3$.