國立臺灣大學105學年度碩士班招生考試試題

(1) (20%) Let G be the group $\langle x, y : x^4 = y^2 = 1, yxy = x^3 \rangle$.

(a) List all different subgroups of G.

(b) Which of them are normal subgroups?

(c) Which pairs of them are isomorphic?

(2) (16%) (a) Let $(G, \cdot)$ be a group and H be a finite subset of G which is closed under the multiplication $\cdot$. Show that H is a subgroup of G.

(b) Let R be a finite ring with identity in which every nonzero element a is cancellable(for any b, c, $ab = ac \Rightarrow b = c$ and $ba = ca \Rightarrow b = c$). Show that R is a division ring.

(3) (12%) Let m and n be positive integers. Determine $\mathrm{Hom}(\mathbb{Z}^m, \mathbb{Z}^n)$, the set of all ring homomorphisms of the ring $\mathbb{Z}^m$ into the ring $\mathbb{Z}^n$. Prove your answer.

(4) (12%) Let $\mathcal{P}$ be the set of all prime numbers and A be the product ring of the fields $\mathbb{Z}/p\mathbb{Z}$, $p \in \mathcal{P}$. Let I be the ideal of A consisting of the elements $(a_p)_{p\in\mathcal{P}}$ such that $a_p \neq 0$ only for finite number of indices p. Let $B = A/I$. Show that, for every positive integer n and every $b \neq 0$ in B, there exists a unique element b' of B such that $nb' = b$.

(5) (16%) Let $R[x]$ be the polynomial ring over a commutative ring R. Show that a polynomial $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is invertible in $R[x]$ if and only if a_0 is invertible in R and $a_1, a_2, \ldots, a_n$ are nilpotent elements in R.

(6) (12%) Let F be a field of characteristic $p \neq 0$, and K be an extension of F. Let $T = \{a \in K : a^{p^n} \in F \text{ for some } n\}$, show that any automorphism of K leaving every element of F fixed also leaves every element of T fixed.

(7) (12%) If a field F contains a primitive n-th root of unity, show that the Galois group of $x^n - a$, for $a \in F$, is abelian.