國立臺灣大學105學年度碩士班招生考試試題

1. (14%) Solve the following ordinary differential equation

$$y''+\frac{1}{x}y'+(1-\frac{1}{4x^2})y=0$$

(A) (2%) Rewrite this ODE into a Quasi-Euler Equation

$$(x-x_0)^2\alpha_0 y''+(x-x_0)\beta_0 y'+\gamma(x)y=0$$

(B) (2%) What is the regular singular point x_0?

(C) (10%) Find the general solution.

2. (10%) Expand the following equation into a Fourier series

$$f(x)=x^2, \quad -\pi\le x\le\pi$$

(A)(2%) Is $f(x)$ an even function or an odd function?

(B) (8%) Find the Fourier series. Check your solution, does it match your answer in question 2(A)?

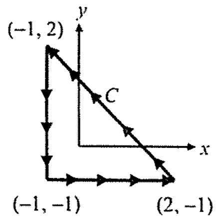
3. (9%) Verify the following Laplace transform

$$\mathrm{L}[at+\sin at]=\frac{a^3}{s^2(s^2+a^2)}$$

4. (8%) Evaluate the integral along the curve C counter-clockwisely

$$\oint_C\left[\left(\frac{-y}{x^2+y^2}+5x^2\right)dx+\left(\frac{x}{x^2+y^2}-6y\right)dy\right]$$

where C is the boundary of a triangle with vertices (-1,-1), (2, -1), (-1,2).



5. (25%)

(A)(20%) Find the solution $u(x,t)$ of the following initial-boundary value problem

$$\begin{cases}\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial u}{\partial t}+4u, & 0<x<1,\ t>0\\ u(0,t)=0, & t>0\\ u(1,t)=\sinh 2, & t>0\\ u(x,0)=\sin 3\pi x, & 0<x<1\end{cases}$$

(B)(5%) Find $u(1/3,t)$ as $t\to\infty$.

6. (34%) Let A, B and C be $n\times n$ matrices.

(A) (4%) Derive the algebraic decomposition:

$$A=A^{(sys)}+A^{(anti)}$$

where $A^{(sys)}$ and $A^{(anti)}$ are respectively symmetric and antisymmetric matrices.

(B) (5%) Denote $\mathrm{trace}(A)=\sum_{i=1}^{n}A_{ii}$. Find $\mathrm{trace}(BC^T)$ where B and C are respectively symmetric and anti-symmetric matrices, and C^T denote the transpose of C. Note $(BC)_{ik}=\sum_{j=1}^{n}B_{ij}C_{jk}$ is a $n\times n$ matrix. Note that BC represents the matrix multiplication of matrices B and C.

(C) (10%) Given B and C being respectively 3×3 symmetric and anti-symmetric matrices, find their associated eigenvalues and eigenvectors (or the governing equations for these eigenvalues and eigenvectors).

(D) (5%) Given B being respectively $n\times n$ symmetric matrix, show that B can be diagonalized.

(E) (5%) Consider the following equation

$$Bx=f$$

where B $n\times n$ is a symmetric matrix, and x and f are respectrively are $n\times 1$ vectors. Solve x (B and f are known) utilizing the diagonalizable property of B.

(F) (5%) The two matrices B and C are called similar if there exists a nonsingular matrix P such that $B=PCP^{-1}$. Show that B and C possess the same eigenvalues. What is the relationship between the eigenvectors of B and C?