1. (8%) Use the Laplace transform to solve the initial value problem $\frac{dy}{dt}+y=f(t); y(0)=0$, where

$$f(t)=\begin{cases}1 & \text{for } 0\le t<5 \\ 2 & \text{for } 5<t\end{cases}$$

2. (32%) Let $\mathbf{A}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}[1,2,3,4]$.

(a) Determine the rank of **A**.

(b) Find a basis for the row space of **A**.

(c) Find a basis for the column space of **A**.

(d) Find the conditions on **b** under which the system $\mathbf{Ax}=\mathbf{b}$ is consistent.

(e) Determine $\det\mathbf{A}$.

(f) The characteristic equation of **A** can be expressed as $p_A(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=\alpha\lambda^m(\lambda-\beta)^n$, where α, $\beta(\neq 0)$, and m are constants. Determine β, m, and n.

(g) Determine the dimension of the subspace spanned by the eigenvectors associated with $\lambda=0$.

(h) Solve the system $\dot{\mathbf{x}}=\mathbf{Ax}$ with $\mathbf{x}(0)=[0,1,1,5]^T$.

3. (12%)

(a) In a 2D rectangular coordinate system, $\overline{\nabla}$ is defined as $\partial/\partial x\hat{i}+\partial/\partial y\hat{j}$, where $\hat{i}$ and $\hat{j}$ are unit vectors in the x and y directions, respectively. Given a scalar field $u(x,y)$, please explain the physical meaning of its gradient, $\overline{\nabla}u$.

(b) Given a vector field $\bar{v}(x,y)=v_x\bar{i}+v_y\bar{j}$, where v_x and v_y are its x and y components, respectively, please explain the physical meaning of its divergence, $\overline{\nabla}\cdot\bar{v}$.

(c) Please explain the physical meaning of the curl of $\bar{v}$, $\overline{\nabla}\times\bar{v}$.

(d) Please offer examples of applications of all three operations above in a scientific or engineering field. Be specific: first explain why a given operation is needed and then demonstrate how it helps analyze the given example.

4. (8%) The 1-D heat equation reads $\alpha^2u_{xx}=u_t$, where $u=u(x,t)$ is the temperature at time t at the point x, and α is the diffusivity of a given material governed by the equation. The solution to its initial-value problem, where the initial temperature $u(x,0)=f(x)$ is given, in an infinite domain $-\infty<x<\infty, 0<t<\infty$, is $u(x,t)=\int_{-\infty}^{\infty}f(\xi)\frac{\exp[-(\xi-x)^2/4\alpha^2t]}{2\alpha\sqrt{\pi t}}d\xi$. Suppose we let $f(x)=\delta(x-x_0)$, where $\delta(x)$ is the Dirac delta function, the solution reduces to $u(x,t)=\frac{\exp[-(x-x_0)^2/4\alpha^2t]}{2\alpha\sqrt{\pi t}}$.

What is the physical meaning of this solution? Is there anything unphysical in this solution? Why?

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5. **(10%)** Consider a set of orthonormal functions $\phi_k(x)$ obtained from a set of linearly independent functions $\varphi_k(x)$ and given by:

$\phi_1(x) = c_{11}\varphi_1(x)$

$\phi_2(x) = c_{21}\varphi_1(x) + c_{22}\varphi_2(x)$

$\phi_3(x) = c_{31}\varphi_1(x) + c_{32}\varphi_2(x) + c_{33}\varphi_3(x)$

.... where c's are constants to be determined.

Please generate a set of polynomials $\phi_k(x)$ orthonormal in the interval (-1, 1) from the sequence $\varphi\{1, x, x^2, x^3, ...\}$ and derive the first three terms $\phi_1(x), \phi_2(x), \text{and}\, \phi_3(x)$. From this orthonormalization process, which type of special function can be obtained?

6. **(8%)** A signal $f(t)$ has its corresponding Fourier spectrum of $F(\omega) = \dfrac{4}{\omega^2 + 4}$. Use the Fourier transform to find $f(t)$, $-\infty < t < \infty$.

7. **(12%)**

(a) Expand the function $f(x) = 1 - x^2, -\dfrac{1}{2} \le x \le \dfrac{1}{2}$ into Fourier series.

(b) From (a), find the series $1 - \dfrac{1}{4} + \dfrac{1}{9} - \dfrac{1}{16} + \dfrac{1}{25} - \cdots\cdots = ?$

(c) Given that the finite Fourier cosine transform of $f(t), 0 < t < \dfrac{1}{2}$ is

$$\tilde{F}_c(n) = \begin{cases} \dfrac{11}{24} & n = 0 \\ \dfrac{(-1)^{n-1}}{4n^2\pi^2} & n = 1, 2, 3, \cdots\cdots \end{cases} \qquad \text{Find } f(t).$$

8. **(10%)** Imagine you worked as a mechanical engineer and were asked to design a spacecraft able to perform atmospheric reentry, that is, entering and travelling through the gases of a planet's atmosphere from outer space (Fig.1). Consider the following questions:

(a) During the later stage (at lower altitudes, as air density increases, depicted schematically from point 1 to point 2 on the trajectory in Fig. 1) of the atmospheric reentry, the spacecraft heats up due to the friction with the atmosphere of the planet. To analyze the temperature distribution of the thermal insulator on the surface of the spacecraft (see inset of Fig. 1), a revised heat equation with a time dependent heat source $q_h = q_h(t)$ is necessary. Please write it down (a derivation is not required) and qualitatively draw $q_h(t)$ from t_1(time at point 1) to t_2 (time at point 2).

(b) To prevent the spacecraft from burning up during the reentry, a cooling system (see inset of Fig. 1) has to be designed. Now, the temperature distribution of the thermal insulator on the spacecraft is governed by the heat equation with a heat source $q_h = q_h(t)$ and a heat sink $q_c = q_c(t)$. Please write it down (a derivation is not required) and qualitatively draw $q_c(t)$ from t_1(time at point 1) to t_2 (time at point 2) based on your own design.

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(c) Please define an initial boundary value problem (IBVP) of a piece of the thermal insulator governed by the heat equation in (b). Briefly explain why your IBVP can help you understand the behavior of the thermal insulator during reentry.

(d) Suppose your IBVP in (c) is too complicated to be solved analytically. What other approaches you may use to tackle it?

(e) Using the new approaches in (d) other than looking for an analytic solution, how do you verify that they do work properly?

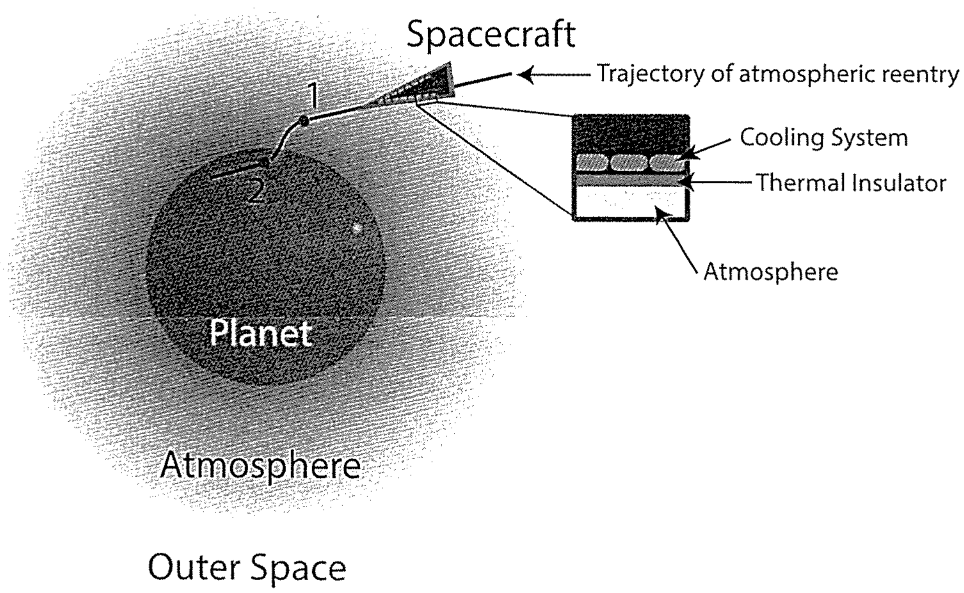

Fig. 1. Schematic of atmospheric reentry.

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