1. 20%

Let $f:[0,1]\to\mathbb{R}$ be a real-valued continuously differential function with $f(0)=0$. Suppose also that there is a constant $M>0$ such that for $0\le x\le 1$, $0\le f'(x)\le M f(x)$. Which of the following statements is true?

A. $f(x)=0$ for $0\le x\le 1$

B. $f(x)>0$ for $0\le x\le 1$

C. $f'(x)>0$ for some $x>0$.

Find and justify your answer.

2. 20%

Consider the vector differential equation

$$\frac{dx(t)}{dt}=A(t)x(t),$$

where A is a smooth $n\times n$ function on $\mathbb{R}$. Assume A has the property that

$$\langle A(t)y,y\rangle\le c\|y\|^2 \quad \text{for all} \quad y\in\mathbb{R}^n,\ t\in\mathbb{R},$$

where c is a fixed real number. Prove that any solution $x(t)$ of the equation satisfies

$$\|x(t)\|\le e^{ct}\|x(0)\| \quad \text{for all } t>0.$$

3. 20%

Let the real-valued function $y(t)$ $(0\le t<\infty)$ solve the initial value problem

$$y''=-|y|,\quad y(0)=1,\quad y'(0)=0.$$

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Which of the following statements is true?

A. $y(t) > 0$ for $t > 0$

B. There is exactly one $t_0 > 0$ such that $y(t_0) = 0$.

C. There exist $t_1 > t_2 > 0$ such that $y(t_j) = 0$ for $j = 1, 2$.

Find and justify your answer.

4. 20%

Let the function $x(t)$ $(-\infty < t < \infty)$ be a solution of the differential equation $\frac{d^2x}{dt^2} - 2b\frac{dx}{dt} + cx = 0$ such that $x(0) = x(1) = 0$, where b and c are real constants. Prove that $x(n) = 0$ for every integer n.

5. 20%

i. Find a basis for the space of real solutions of the differential equation

$$(*) \qquad \sum_{n=0}^{7} \frac{d^n x}{dt^n} = 0$$

ii. Find a basis for the subspace of real solutions of (*) that satisfy $\lim_{t \to \infty} x(t) = 0$.

試題隨卷繳回