國立臺灣大學 104 學年度碩士班招生考試試題

1. (35%)
   (a) Show that if $|G| = 240$ and $H \leq G$ with $H \cong A_5$, then $H \lhd G$.
   (b) Show that if G is a p-group where p is a prime integer (i.e. $|G| = p^n, n \in \mathbb{N}$) , then the center $Z(G)$ of G is not a trivial subgroup and in particular, if G is a non-abelian group of order p^3, then $|Z(G)| = p$.
   (c) Classify groups of order 45. (Justify your answers.)

2. (30 %) (Justify your answers)
   (a) Find all $c \in \mathbb{Z}_5$ such that $\mathbb{Z}_5[x]/\langle x^2 + cx + 1\rangle$ is a field.
   (b) Determine all ideals of the ring $\mathbb{Z}[x]/\langle 5, x^3 - x^2 + x + 4\rangle$.
   (c) Prove or disprove the following statements.
   (1) The polynomial ring $\mathbb{R}[x, y]$ in two variables is a Euclidean domain.
   (2) The polynomial ring $\mathbb{R}[x]$ in one variable is a PID.

3. (a) (17%)
   (1) Let $L : K$ be an extension and $\alpha \in L$ be algebraic over K. Show that if α is algebraic over K then $K[\alpha] = K(\alpha)$.
   (2) Show that $x^3 - 3x + 3$ is irreducible in $\mathbb{Q}[x]$. Suppose that α is a root of $x^3 - 3x + 3$ in $\mathbb{C}$. Find the minimal polynomial of $\beta = 1 - \alpha + \alpha^2$.

   (b) (18%) Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$.
   (1) Find a primitive generator for L over $\mathbb{Q}$. Explain your answer in detail.
   (2) Show that $L/\mathbb{Q}$ is a Galois extension.
   (3) Determine $\mathrm{Gal}(L/\mathbb{Q})$, lattice of subgroups of this group and the corresponding intermediate fields between L and $\mathbb{Q}$.

試題隨卷繳回