國立臺灣大學 104 學年度碩士班招生考試試題

※ 注意：請於試卷內之「非選擇題作答區」依序作答，並應註明作答之大題及小題題號。

1. (20%) Label the following statements as being true or false. *(No explanation is needed. Each correct answer gets 2% and each wrong answer gets 0%):*
   (a) Let R be the reduced row echelon form of A. If $PA = R$, then P is invertible.
   (b) Let $\mathcal{S}_1$ be a linearly independent subset of $\mathcal{R}^n$ and $\mathcal{S}_2$ be a generating set for $\mathcal{R}^n$. Then the number of vectors in $\mathcal{S}_1$ is less than equal to that in $\mathcal{S}_2$.
   (c) Let A be an $n \times n$ matrix. If $A\mathrm{v}_i = \lambda_i \mathrm{v}_i$ for n linearly independent vectors v_i, then A is diagonalizable.
   (d) If the column vectors of BA are linear independent, then the column vectors of A are linearly independent.
   (e) Let A be a 3×4 matrix and $\mathrm{b} \in \mathcal{R}^3$. Then $A\mathrm{x} = \mathrm{b}$ has infinitely many solutions.
   (f) Let A be an $m \times n$ matrix. Then there exists a linear transformation $T : \mathcal{R}^n \to \mathcal{R}^m$ with standard matrix A.
   (g) Let C^∞ be the subset of $\mathcal{F}(\mathcal{R})$ consisting of all those functions that have derivatives of all orders. Let T be a linear operator on C^∞. Then T is one-to-one if and only if T is onto.
   (h) Let W be a subspace of a vector space V and Span $\mathcal{S} = W$. Then $\mathcal{S}^\perp = W^\perp$.
   (i) If two $n \times n$ matrices have the same characteristic polynomials, then they are similar.
   (j) The reduced row echelon form of A^{-1} is an identity matrix.

2. (10%) Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$, $V = \text{Span } \mathcal{B}$, and T be a linear operator on V defined by
$$T(A) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} A \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$
   Find $[T]_\mathcal{B}$, the matrix representation of T with respect to $\mathcal{B}$.

3. Let A be a 4×4 matrix with $\det A = 2$. (a) (5%) Find the reduced row echelon form of $[-2A \;\; A]$. (b) (5%) Find $\det \begin{bmatrix} -2A^{-1} & A \\ 0 & A^2 \end{bmatrix}$.

4. Let $W = \text{Span } \{\mathrm{v}_1, \; \mathrm{v}_2\}$ where
$$\mathrm{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathrm{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$
   (a) (4%) Find a basis for $W^\perp$.
   (b) (4%) Find an orthogonal basis for W.
   (c) (2%) Find the intersection of W and $W^\perp$, i.e., $W \cap W^\perp$.

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5. (24%) A total of n, $n > 0$, nodes are placed on a 2D plane. To determine the links between these nodes, sequential experiments of coin tosses are conducted as follows: for each pair of nodes, a coin is randomly chosen from a box of a sufficiently large number of coins and then tossed. If the head appears, a link is drawn to connect the two nodes. After the toss, the coin is returned to the box and the experiment repeats for the next pair. Note that due to production problems at the mint, all coins in the box are biased. The probability of head is different for different coins, but it can be assumed to follow a uniform distribution over the interval $[0.4, 0.8]$.

   (a) (6%) The set of nodes and the links thus drawn between them constitute a random graph. Let X be the number of links in the graph. What is $Var[X]$?

   (b) (6%) An isolated node is formed if it is not connected to any other nodes in the graph. Let Y be the number of isolated nodes in the graph. What is $E[Y]$?

   (c) (3%) A graph is said to be connected if no node is isolated. Let $p(n)$ be the probability that a random graph of n nodes is connected, where $p(1) = 1$ by definition. Use (b) to find a lower bound for $p(n)$.

   (d) (6%) While it is difficult to obtain the general form of $p(n)$ directly, it can be obtained recursively as follows:

$$p(n) = 1 - \sum_{i=1}^{n-1} f(n,i)p(i), \quad \forall n \geq 2,$$

   where $f(n,i)$ is a function of n and i. What is $f(n,i)$?

   (e) (3%) Find $p(4)$.

6. (26%) In a multi-path channel with n paths, the received signal can be represented as

$$Re^{j\phi} = \sum_{i=1}^{n} A_i e^{j\theta_i} = \sum_{i=1}^{n} X_i + j\sum_{i=1}^{n} Y_i,$$

where R and ϕ are the magnitude and phase of the received signal respectively, and A_i and θ_i are the magnitude and phase of the i^{th} multi-path component respectively. In a channel rich of paths, $\sum_{i=1}^{n} X_i$ and $\sum_{i=1}^{n} Y_i$ can be assumed to be independent zero-mean Gaussian random variables with variance equal to σ^2.

   (a) (8%) Find the probability density functions (PDFs) of R and ϕ.

   (b) (6%) The receiver is unable to detect the desired signal if the instantaneous power $R^2 < \gamma$, where γ is a threshold. Find the PDF of R^2 and the probability that the received signal cannot be detected.

   (c) (6%) A total of N, $N \geq 1$, jammers are present to send interfering signals through the channel. The interfering signals $\{I_i \,|\, 1 \leq i \leq N\}$ can be considered as *iid* random variables with the same PDF as the desired signal but with smaller mean power such that $E[I_i^2] = \frac{1}{\beta}E[R^2]$, $\beta > 1$. The exact value of N is unknown, but it is known that N follows a geometric distribution with expected value q. Let $I^2 = \sum_{i=1}^{N} I_i^2$ be the sum of powers of all interfering signals. What is the PDF of I^2?

   (d) (6%) The jammers are considered to successfully jam the desired signal if $\frac{R^2}{I^2} < \eta$, where η is a threshold. What is the probability that this occurs?