國立臺灣大學 103 學年度碩士班招生考試試題

Probability Theory and Statistics (50%)

1. Consider throwing a fair die, where the sample space S = {⚀, ⚁, ⚂, ⚃, ⚄, ⚅}.

(a) Please define any two different random variables, say, X and Y, based on S and write down the probability mass functions $P_X(x)$ and $P_Y(y)$ respectively. (6%)

(b) Let event A={⚀, ⚁, ⚄}. Derive $P_{Y|A}(y)$. (4%)

2. Alex, Ben, and Tim are three prisoners, one of whom is sentenced to die with equal probability. The three prisoners cannot communicate with each other. Alex asks a jailer to tell him who will be freed so that Alex could ask him to bring a letter to Alex's wife. The jailer tells Alex that Ben is going to be freed. So, Alex considers the probability of himself dying as 1/2 after learning this information. Is Alex correct? Why or why not? Analyze and write down your **quantitative reasoning** based on the notion of conditional probability and independence. (10%)

3. Given

$$f_{XY}(x,y)=\begin{cases} x+y, & 0<x<1,\ 0<y<1; \\ 0, & \text{otherwise.} \end{cases}$$

(a) Are X and Y independent? (3%)

(b) Derive the correlation coefficient $\rho(X,Y)$ = ? (5%)

(c) Derive $E[X|Y=y]$. (3%)

(d) $Z = X-Y$. Derive f_Z. (4%)

4. You are a sport agent and are comparing the performance of two baseball pitchers, W and P. The speed of a strike pitch by a pitcher is

$$Y_{ik} = \theta_i + \omega_{ik},\ i \in \{W,P\} \text{ and } k=1, 2, 3, \ldots$$

where θ_i is the mean speed of pitcher i, an unknown constant, and ω_{ik} represents the variation in speed of pitcher i, and is $N(0, \sigma^2)$ and independent and identical over time index k and between the two pitchers.

(a) Given 11 speed measurements of Y_{ik} as listed in the table below for each pitcher, design an estimator of θ_i ? Is your estimate biased or unbiased, why? (8%)

	1	2	3	4	5	6	7	8	9	10	11
W	91	89	93	94	92	88	91	90	90	93	89
P	95	90	87	92	96	94	90	92	95	96	88

(b) Does pitcher W pitch faster than pitcher P? Please explain how you apply statistical methods to reach a conclusion and what your level of confidence is. (7%)

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Linear Algebra (50%)

5. Let A be an $m \times n$ matrix. Let B be a matrix obtained by permuting two columns of A. Please indicate whether the following statements are true or false. (No proof is needed.)
(a) (2%) $\text{rank}(A) = \text{rank}(B)$.
(b) (2%) The column space of A is the same as the column space of B.
(c) (2%) The row space of A is the same as the row space of B.
(d) (2%) If $m > n$, then the dimension of the column space of A can not be equal to m.
(e) (2%) If $m = n$, then the determinant of A equals the determinant of B.

6. Let A be an $m \times n$ matrix. Let $\mathbf{x} = [x_1, x_2, ..., x_n]^t$ and $\mathbf{b} = [b_1, b_2, ..., b_m]^t$ be $n \times 1$ matrix and $m \times 1$ matrix respectively. Suppose that the system of linear equations represented by $A\mathbf{x} = \mathbf{b}$ has at most one solution for any $\mathbf{b} \in \mathbf{R}^m$. Please indicate whether the following statements are true or false. (No proof is needed.)
(a) (2%) The nullity of A is 1.
(b) (2%) $\text{rank}(A) = m$.
(c) (2%) The rows of A are linearly independent.
(d) (2%) The columns of A are linearly independent.
(e) (2%) Suppose that A is a matrix representation of $T : \mathrm{R}^n \to \mathrm{R}^m$ for some bases. Then, T is one-to-one.
(f) (2%) Let R be the reduced row echelon form of A. Then, the column space of A is identical to the column space of R.

7. Let A and B be $n \times n$ matrices. Suppose that A is diagonalizable. Please indicate whether the following statements are true or false. (No proof is needed.)
(a) (2%) If $B = P^{-1}AP$ for an invertible $n \times n$ matrix P, then B is diagonalizable.
(b) (2%) If A and B have the same characteristic polynomials, then B is also diagonalizable.
(c) (2%) If B is diagonalizable, then $A + B$ is diagonalizable.

8. Let $T : \mathrm{M}^{3\times3} \to \mathrm{M}^{3\times3}$ be a linear operator defined by $T(A) = \frac{A+A^t}{2}$ for $A \in \mathrm{M}^{3\times3}$, where $\mathrm{M}^{3\times3}$ is the set of all 3×3 matrices with real entries .
(a) (8%) What are the eigenvalues and the associated eigenspaces of T?
(b) (4%) Show a matrix representation of T which is diagonal.

9. Let T be a linear operator on a real inner product space V. Suppose that $\|T(\mathbf{x})\| = \|a\mathbf{x}\|$ for all $\mathbf{x} \in V$, where a is a real number.
(a) (5%) Find the values of a so that T is one-to-one.
(b) (5%) Find the value of b so that $< T(\mathbf{x}), T(\mathbf{y}) > = b < \mathbf{x}, \mathbf{y} >$ for any $\mathbf{x}$ and any $\mathbf{y}$ in V.

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